Kannur University

Department of Mathematical Sciences

MSMAT03 MDC1

Calculus

with an introduction to linear algebra

Problem sheet

A learning guide with worked examples and practice

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Starting point. Plus Two mathematics: elementary algebra, coordinates, factorisation, and basic trigonometry. New ideas are introduced before they are used. Throughout, scalars are real numbers and trigonometric angles are in radians.

How to use each sheet. Read the definitions, work through the examples with a pencil, then cover the solutions and try the three exercises. Compare your method with the answers and hints at the end. For a graph or a theorem question, include the reasoning as well as the final answer.

The route. We start with sets and functions, build integration from rectangles and step functions, then study continuity and differentiation. The last module develops the basic language and calculations of linear algebra.

Module	Main questions
I	How can sums of rectangles define an integral? How do we calculate areas?
II	What do limits and continuity tell us about values, roots and inverses?
III	How do derivatives measure change and describe the shape of a graph?
IV	How do vectors, matrices and linear maps describe linear problems?

Scope. These are selected foundational problem sheets based on the four modules in the supplied syllabus, rather than a complete textbook or a replacement for the full syllabus. Longer general proofs and advanced linear-algebra constructions are left for lectures and further reading.

Text to read alongside. Tom M. Apostol, *Calculus, Volume I: One-Variable Calculus, with an Introduction to Linear Algebra*, second edition, Wiley. The integration-first order follows that text. The explanations and problems here are written for this guide; they are not reproduced textbook exercises. Publisher information: `wiley.com`, Calculus, Volume I, second edition.

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Module I Integral calculus

Sets and functions lead to rectangles, step functions, upper and lower sums, and finally integrals and areas.

I.1 Sets, Cartesian products and intervals

The language we need

A **set** is a collection of objects. Write $x \in A$ when x belongs to A, and $A \subseteq B$ when every member of A belongs to B. $A \cup B$ contains members of either set; $A \cap B$ contains common members. The empty set is $\varnothing$. The real numbers form the set $\mathbb{R}$.

An **ordered pair** (x, y) records the first and second coordinates in that order. The **Cartesian product** is

$$A \times B = \{(x, y) : x \in A,\ y \in B\}.$$

For $a < b$, the interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ includes both endpoints, whereas $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ excludes both. The intervals $[a, b)$ and $(a, b]$ include just one endpoint. An infinite endpoint is always written with a round bracket: $[2, \infty)$.

Example 1. Listing a Cartesian product

Let $A = \{0, 2\}$ and $B = \{1, 3\}$. Find $A \times B$.

Solution. Pair each member of A with every member of B:

$$A \times B = \{(0, 1), (0, 3), (2, 1), (2, 3)\}.$$

There are $2 \cdot 2 = 4$ pairs. Here $(0, 1)$ belongs to $A \times B$, but $(1, 0)$ does not. In general, reversing a pair changes it.

Example 2. Combining intervals

Let $A = [-2, 2)$ and $B = (0, 4]$. Find $A \cap B$ and $A \cup B$.

Solution. The common points satisfy $0 < x < 2$, so $A \cap B = (0, 2)$. Taken together, the intervals contain every number from -2 to 4, including both endpoints. Thus $A \cup B = [-2, 4]$. Check the endpoints separately: $0 \notin B$ and $2 \notin A$.

Try these

1. For $A = \{-1, 1\}$ and $B = \{0, 2\}$, list $A \times B$ and $B \times A$.

2. Write $-1 < x \leq 3$ in interval notation. Does the resulting interval contain -1, 0 and 3?

3. Find $[-1, 3] \cap (1, 5)$ and $[-1, 3] \cup (1, 5)$.

I.2 Functions, domains and graphs

A function assigns one output to each input

A **function** $f : A \to B$ assigns exactly one number $f(x) \in B$ to each $x \in A$. The set A is its **domain**; B is the **codomain**; the **range** is the set of outputs actually obtained. Its graph is $\{(x, f(x)) : x \in A\} \subseteq A \times B$. For a formula with no stated domain, take all real inputs for which the formula is defined: a denominator cannot be zero and a real square root needs a nonnegative expression underneath it. A piecewise formula uses different rules on different parts of its domain.

Example 1. A restricted quadratic

Let $f : [-1, 2] \to \mathbb{R}$ be $f(x) = x^2$. Find $f(-1)$, $f(2)$ and the range.

Solution. Substitution gives $f(-1) = 1$ and $f(2) = 4$. Squares are nonnegative, the minimum 0 occurs at $x = 0$, and every value between 0 and 4 occurs on $[0, 2]$. Thus the range is $[0, 4]$. The codomain is still $\mathbb{R}$. The points $(-1, 1)$ and $(1, 1)$ are allowed: different inputs may have the same output.

Example 2. Finding a domain

Find the real domain of $g(x) = \sqrt{3 - x}/(x + 1)$.

Solution. The square root requires $3 - x \geq 0$, or $x \leq 3$. The denominator requires $x \neq -1$. Both conditions must hold, so the domain is

$$(-\infty, -1) \cup (-1, 3].$$

The endpoint 3 is allowed because the numerator may be zero.

Example 3. Reading a piecewise formula

Let $h(x) = \begin{cases} x + 1, & x < 0, \\ 2, & x \geq 0. \end{cases}$ Find $h(-2)$, $h(0)$ and $h(3)$.

Solution. At -2 use the first rule: $h(-2) = -1$. At 0 and 3 use the second rule: both outputs are 2. At the join, the inequality $x \geq 0$ tells us which rule includes 0. To sketch it, draw $y = x + 1$ for $x < 0$ with an open circle at $(0, 1)$, and $y = 2$ for $x \geq 0$ with a filled circle at $(0, 2)$.

Try these

1. For $f : [-2, 1] \to \mathbb{R}$, $f(x) = x^2 + 1$, find $f(-2)$, $f(0)$ and the range.

2. Find the real domain of $\sqrt{x + 2}/(x - 1)$.

3. For $p(x) = \begin{cases} 2x, & x < 1, \\ x + 3, & x \geq 1, \end{cases}$ find $p(0)$, $p(1)$ and $p(2)$. Sketch the graph, marking the included and excluded endpoints at $x = 1$.

I.3 Partitions and step functions

Cutting an interval into pieces

A **partition** of $[a, b]$ is a finite list $P = \{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$. The ith width is $\Delta x_i = x_i - x_{i-1}$. A **step function** has a constant value c_i on each open interval (x_{i-1}, x_i). Its values at the finitely many partition points may be specified separately. These endpoint values will not affect its integral. A **refinement** adds partition points. To add or multiply two step functions, first use the common refinement containing both sets of breakpoints.

Example 1. Reading widths and heights

On $[0, 4]$, let s have heights $2, -1, 3$ on $(0, 1), (1, 3), (3, 4)$, respectively. Describe a suitable partition.

Solution. Take $P = \{0, 1, 3, 4\}$. The widths are $1 - 0 = 1$, $3 - 1 = 2$ and $4 - 3 = 1$. The corresponding heights are $2, -1, 3$. The middle rectangle has width 2, even though it is just one partition piece. The heights and widths play different roles.

Example 2. Adding and multiplying with different breakpoints

On $[0,3]$, let $s=1$ on $(0,1)$ and $s=2$ on $(1,3)$. Let $t=3$ on $(0,2)$ and $t=-1$ on $(2,3)$. Find $s+t$ and st away from the breakpoints.

Solution. Use $\{0,1,2,3\}$, which includes both breakpoints. Read both functions on each smaller interval:

interval	s	t	$s+t$	st
$(0,1)$	1	3	4	3
$(1,2)$	2	3	5	6
$(2,3)$	2	-1	1	-2

Every new entry is obtained by ordinary addition or multiplication of heights. If values at a breakpoint are given, add or multiply those values separately.

Try these

1. For the partition $\{-1,0,2,5\}$, list the three subinterval widths.

2. A function is 4 on $(0,2)$ and -2 on $(2,5)$. Express its heights using the refined partition $\{0,1,2,4,5\}$.

3. On $[0,3]$, let s have heights $2,0$ on $(0,1),(1,3)$, and let t have heights $1,3$ on $(0,2),(2,3)$. Give the heights of $s+t$ and st on a common refinement.

I.4 Integrating a step function

Signed rectangles define the first integral

If s has height c_i on (x_{i-1},x_i), define

$$\int_a^b s(x)\,dx=\sum_{i=1}^{n} c_i\Delta x_i.$$

Here $\sum$ means "add all the terms", x is the input variable, and dx records that we integrate with respect to x. Positive heights contribute positively and negative heights negatively. The ordinary geometric area between the graph and the axis is instead $\sum_i |c_i|\,\Delta x_i$. Splitting a rectangle does not change its contribution. Thus refining the partition or changing finitely many endpoint values leaves the integral unchanged.

Example 1. Integral and geometric area

For the step function in Example 1 of Sheet I.3, calculate the integral and area.

Solution. Multiply each height by its own width:

$$\int_0^4 s(x)\,dx=2(1)+(-1)(2)+3(1)=3.$$

For area, make each rectangle contribution nonnegative: $2(1)+1(2)+3(1)=7$ square units. The integral subtracts the area below the axis.

Example 2. Refinement and endpoint values

Let $s(x) = 2$ for $0 < x < 3$, but $s(0) = 100$ and $s(3) = -100$. Find its integral using $\{0, 1, 3\}$.

Solution. On the two open pieces the height is 2, so

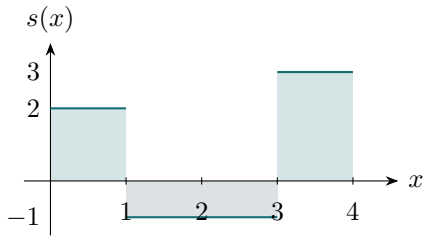
$$\int_0^3 s(x)\,dx = 2(1-0) + 2(3-1) = 2 + 4 = 6.$$

The same answer comes from one rectangle of width 3. A value at an isolated point adds no rectangle of positive width, so the two unusual endpoint values make no contribution.

Example 3. Linearity in a calculation

Suppose s has heights $1, 3$ and t has heights $2, -1$ on $(0, 1), (1, 2)$. Find $\int_0^2 (2s - t)\,dx$.

Solution. The heights of $2s - t$ are $2(1) - 2 = 0$ and $2(3) - (-1) = 7$. Both widths are 1, giving an integral of 7. Equivalently, $\int s = 4$ and $\int t = 1$, so $2\int s - \int t = 8 - 1 = 7$. This illustrates **linearity**: constants and sums can be taken outside the step-function integral.



The three rectangles in Example 1; the middle contribution is negative.

Try these

1. A step function has heights $3, -2$ on $(0, 2), (2, 5)$. Find its integral on $[0, 5]$ and the geometric area between its graph and the axis.

2. Let $s = 4$ on $(1, 2)$ and $s = 1$ on $(2, 4)$, with arbitrary endpoint values. Calculate $\int_1^4 (3s - 2)\,dx$.

3. A step function has height k on $(0, 2)$ and height 1 on $(2, 3)$. If its integral on $[0, 3]$ is 9, find k.

I.5 Upper sums, lower sums and integrability

Trapping a function between rectangles

Let f be **bounded** on $[a, b]$: its values stay between two finite numbers. On each closed partition piece $[x_{i-1}, x_i]$, let m_i be the infimum (greatest lower bound) and M_i the supremum (least upper bound) of its values. When a smallest or largest value exists, it is the infimum or supremum. Define the **lower** and **upper sums**

$$L(f, P) = \sum_i m_i \Delta x_i, \qquad U(f, P) = \sum_i M_i \Delta x_i.$$

Refinement can raise the lower sum and lower the upper sum. The **lower integral** is $\sup_P L(f, P)$; the **upper integral** is $\inf_P U(f, P)$. The function is **Riemann integrable** when these are equal; the common value is $\int_a^b f(x)\,dx$. Equivalently, we can make $U(f, P) - L(f, P)$ as small as we wish. For an integrable function, every partition gives $L(f, P) \le \int_a^b f \le U(f, P)$. These sums extend the step-function idea: smaller and larger step rectangles trap the graph. Unlike a step-function integral, a sum for a general f can depend on the partition.

Example 1. Lower and upper sums for an increasing function

Let $f(x) = x$ on $[0, 1]$. Use n equal pieces to find its integral.

Solution. Each width is $1/n$. Since f increases, the smallest value on the ith piece is $(i-1)/n$, and the largest is i/n. Thus

$$L_n = \frac{0 + 1 + \cdots + (n-1)}{n^2} = \frac{n-1}{2n}, \qquad U_n = \frac{1 + 2 + \cdots + n}{n^2} = \frac{n+1}{2n}.$$

We used $1 + \cdots + n = n(n+1)/2$. The gap is $1/n$, which tends to 0, and both bounds tend to $1/2$. Hence $\int_0^1 x\,dx = 1/2$. For four pieces, the bounds are $3/8$ and $5/8$; these are approximations, not the exact integral.

Why monotone functions are integrable

A function is **nondecreasing** if $x \le y$ implies $f(x) \le f(y)$, and **nonincreasing** if it implies $f(x) \ge f(y)$. Either kind is called **monotone**. On a closed interval a real-valued monotone function is bounded by its endpoint values. For an equal-width partition, a nondecreasing function satisfies

$$U_n - L_n = \frac{b-a}{n}\left[(f(x_1) - f(x_0)) + \cdots + (f(x_n) - f(x_{n-1}))\right] = \frac{(b-a)(f(b) - f(a))}{n}.$$

The intermediate terms cancel. For a nonincreasing function the same formula uses $f(a) - f(b)$. The gap tends to 0, proving integrability. A bounded function monotone on each of finitely many subintervals is also integrable.

Example 2. Decreasing functions use the opposite endpoints

For $f(x) = 3 - x^2$ on $[0, 1]$, find the two-piece bounds and a bound for the gap with n equal pieces.

Solution. On this interval x^2 increases, so f decreases. With endpoints $0, 1/2, 1$, right endpoints give the lower sum:

$$L_2 = \tfrac{1}{2}(\tfrac{11}{4} + 2) = \tfrac{19}{8}, \qquad U_2 = \tfrac{1}{2}(3 + \tfrac{11}{4}) = \tfrac{23}{8}.$$

Monotonicity proves integrability, and $19/8 \le \int_0^1 f \le 23/8$. For n pieces the gap is $(1/n)(3-2) = 1/n$. Thus $n = 101$ makes the gap less than 0.01, without evaluating the integral.

Try these

1. For $f(x) = x$ on $[0, 2]$ with four equal pieces, calculate L_4 and U_4.

2. For $f(x) = x^2$ on $[0, 1]$, show that the gap with n equal pieces is $1/n$. Give an integer n making it less than $1/20$.

3. Explain why $f(x) = |x|$ is integrable on $[-1, 1]$, although it is not monotone on the whole interval. State the two intervals to use.

I.6 Polynomial integrals and basic properties

Rules for integrable functions

If f, g are integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

For $a < c < b$, $\int_a^b f = \int_a^c f + \int_c^b f$. Define $\int_b^a f = -\int_a^b f$ and $\int_a^a f = 0$. If $f \le g$, then $\int_a^b f \le \int_a^b g$ for $a \le b$. In particular, $m \le f \le M$ gives $m(b-a) \le \int_a^b f \le M(b-a)$. The variable name is immaterial: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

Every polynomial is integrable. For an integer $r \geq 0$,

$$\boxed{\int_a^b x^r \, dx = \frac{b^{r+1} - a^{r+1}}{r+1}.}$$

For $r = 0$ this is the constant function 1, whose integral is $b - a$. We use the formula and linearity; differentiation is not needed here.

Example 1. Seeing the square rule from rectangle sums

Show that $\int_0^b x^2 \, dx = b^3/3$ for $b > 0$.

Solution. Use n equal pieces of width b/n. Since x^2 increases on this interval,

$$U_n = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 = b^3 \frac{n(n+1)(2n+1)}{6n^3} \longrightarrow \frac{b^3}{3}.$$

Here we used the finite-sum identity $1^2 + \cdots + n^2 = n(n+1)(2n+1)/6$. The lower sum uses $i = 0, \ldots, n-1$; its gap from U_n is b^3/n, which tends to zero. Both sums therefore have the same limit $b^3/3$. The general polynomial rule has a similar rectangle-sum justification.

Example 2. A polynomial on an interval crossing zero

Calculate $\int_{-1}^{2} (3x^2 - 2x + 1) \, dx$.

Solution. Integrate each term using the displayed rule:

$$\begin{aligned} \int_{-1}^{2} (3x^2 - 2x + 1) \, dx &= 3\frac{2^3 - (-1)^3}{3} - 2\frac{2^2 - (-1)^2}{2} + (2 - (-1)) \\ &= 9 - 3 + 3 = 9. \end{aligned}$$

Brackets around a negative endpoint prevent sign errors.

Example 3. Using properties when the formula is unknown

An integrable f satisfies $\int_0^3 f = 7$ and $\int_0^1 f = 2$. Find $\int_1^3 (2f(x) + 1) \, dx$ and $\int_3^1 f$.

Solution. Interval addition gives $\int_1^3 f = 7 - 2 = 5$. Linearity then gives

$$\int_1^3 (2f + 1) = 2(5) + (3 - 1) = 12.$$

Reversing the endpoints gives $\int_3^1 f = -5$. No explicit formula for f was required.

Try these

1. Calculate $\int_0^2 (2x^2 + 3x - 1) \, dx$ and $\int_1^0 x^3 \, dx$.

2. If $\int_0^4 f = 9$ and $\int_0^1 f = 3$, calculate $\int_1^4 (2f - 1)$.

3. An integrable function satisfies $2 \leq f(x) \leq 5$ on $[1, 4]$. Give lower and upper bounds for its integral. Can its integral equal 16?

I.7 Area under and between curves

Draw the region before integrating

For an integrable nonnegative f on $[a, b]$, the region $\{(x, y) : a \le x \le b,\ 0 \le y \le f(x)\}$ is its **ordinate set**; its area is $\int_a^b f(x)\,dx$. More generally, if integrable functions satisfy $g(x) \le f(x)$ throughout $[a, b]$, the area between them is

$$A = \int_a^b (f(x) - g(x))\,dx.$$

First find intersections, then decide which graph is on top. If the order changes, split the interval at the crossing points. Area is always nonnegative. An integral of a signed function may be negative or zero even when the region has positive area.

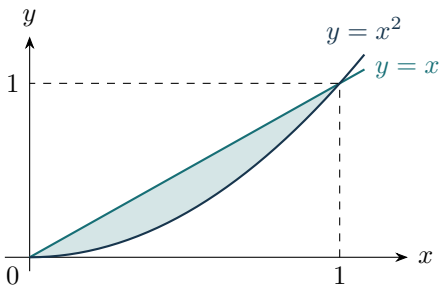
Example 1. A line and a parabola

Find the area enclosed by $y = x$ and $y = x^2$.

Solution. Solve $x = x^2$, giving $x = 0, 1$. For $0 \le x \le 1$, $x - x^2 = x(1 - x) \ge 0$, so the line is on top. Therefore

$$A = \int_0^1 (x - x^2)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

The width is measured along the x-axis and each thin rectangle has height $x - x^2$.



Example 2. The two curves exchange order

Find the area between $y = x$ and $y = 0$ over $[-1, 1]$.

Solution. The graphs cross at $x = 0$. The axis is above the line on $[-1, 0]$, and the line is above the axis on $[0, 1]$. Hence

$$A = \int_{-1}^{0} (0 - x)\,dx + \int_0^1 (x - 0)\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

In contrast, $\int_{-1}^{1} x\,dx = 0$ because the signed contributions cancel.

Try these

1. Find the area under $y = x^2$ and above the axis on $[0, 2]$.

2. Find the area enclosed by $y = 2x$ and $y = x^2$. State the intersections and which graph is above the other.

3. Find the area between $y = x - 1$ and the axis on $[0, 3]$. Split the calculation where the line crosses the axis.

Module II Continuous functions

Limits describe nearby values; continuity lets us pass from nearby values to roots, inverse functions, and integrals.

II.1 Limits: what happens near a point?

Meaning and notation

The statement $\lim_{x\to a} f(x) = L$ says that $f(x)$ can be made as close as we wish to L by taking x sufficiently close to a, with $x \neq a$. The value $f(a)$ need not exist. More precisely, for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon$$

for x in the domain. Here ε is the allowed error in the output and δ is a sufficient allowance in the input.

The left limit uses $x < a$; the right limit uses $x > a$. At an interior point a two-sided limit exists exactly when both one-sided limits exist and agree. Limits respect sums, products, and quotients when the limiting denominator is nonzero. For polynomials, substitute $x = a$. If a quotient gives $0/0$, simplify first: $0/0$ is not an answer.

Example 1. Cancel first, then take the limit

Find $\lim_{x\to 2} \dfrac{x^2 - 4}{x - 2}$.

Solution. For $x \neq 2$,

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2.$$

Thus the limit is 4. We did not assign a value to the original quotient at 2. To check the precise definition, observe that $|(x + 2) - 4| = |x - 2|$. Given $\varepsilon > 0$, choose $\delta = \varepsilon$; the required implication follows.

Example 2. Remove a square root by rationalising

Find $\lim_{x\to 0} \dfrac{\sqrt{1 + x} - 1}{x}$.

Solution. Multiply numerator and denominator by the conjugate:

$$\frac{\sqrt{1 + x} - 1}{x} = \frac{(1 + x) - 1}{x(\sqrt{1 + x} + 1)} = \frac{1}{\sqrt{1 + x} + 1} \quad (x \neq 0).$$

The new denominator tends to 2, so the limit is $1/2$.

Example 3. Check both sides

Does $\lim_{x\to 0} \dfrac{|x|}{x}$ exist?

Solution. If $x > 0$, the quotient is 1. If $x < 0$, it is -1. The right limit is 1 and the left limit is -1. They disagree, so the two-sided limit does not exist.

Try these

1. Find $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$. For the simplified expression, give a choice of δ for a prescribed $\varepsilon > 0$.

2. Find $\lim_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}$.

3. Let $p(x) = x+1$ for $x < 1$, and $p(x) = 3-x$ for $x \geq 1$. Find the left and right limits at 1. Does the two-sided limit exist?

II.2 Continuity: matching the value and the limit

Definition and useful rules

A function f is **continuous at** a when $f(a)$ is defined and

$$\lim_{x\to a} f(x) = f(a).$$

At an endpoint of its domain, take the limit from within the domain. A function is continuous on an interval if it is continuous at every point there. Polynomials are continuous on $\mathbb{R}$; a rational function is continuous wherever its denominator is nonzero. The square-root function is continuous on $[0,\infty)$.

Sums and products of continuous functions are continuous. A quotient is continuous where its denominator is nonzero. If f is continuous at a and g is continuous at $f(a)$, the composite $(g\circ f)(x) = g(f(x))$ is continuous at a. Always determine its domain first.

Example 1. Fill a hole continuously

For $x \neq 1$ set $f(x) = (x^2-1)/(x-1)$, and set $f(1) = k$. Find k so that f is continuous at 1.

Solution. Away from 1, factor and cancel to get $f(x) = x+1$. Therefore $\lim_{x\to 1} f(x) = 2$. Continuity requires $f(1) = 2$, so $k = 2$. With this choice $f(x) = x+1$ for every real x.

Example 2. A composite function and its domain

Where is $h(x) = \sqrt{4-x^2}$ continuous?

Solution. Real square roots require $4-x^2 \geq 0$, or $-2 \leq x \leq 2$. The inner function $4-x^2$ is a polynomial, and the outer square root is continuous for nonnegative inputs. Hence h is continuous on its whole domain $[-2,2]$, using one-sided continuity at the endpoints. Its graph is the upper semicircle of radius 2.

Example 3. Match two pieces

Let $q(x) = 2x+k$ for $x < 1$, and $q(x) = x^2+2$ for $x \geq 1$. Find k for continuity everywhere.

Solution. Each polynomial piece is continuous on its own interval. Only the joining point needs checking. The left limit at 1 is $2+k$; the right limit and the value are both 3. Thus $2+k = 3$, giving $k = 1$.

Try these

1. For $x \neq 2$, let $f(x) = (x^2-4)/(x-2)$. What value should be assigned to $f(2)$ to make f continuous?

2. Find the real domain of $g(x) = \sqrt{3-x}$ and state where it is continuous. Also state where $r(x) = 1/(x+2)$ is continuous.

3. Let $q(x) = kx+1$ for $x < 2$, and $q(x) = x^2-1$ for $x \geq 2$. Find k for continuity on $\mathbb{R}$.

II.3 Continuous functions take intermediate values

Intermediate-value and Bolzano theorems

If f is continuous on $[a, b]$, it takes every value between $f(a)$ and $f(b)$. Thus, if N lies strictly between these endpoint values, there is a $c \in (a, b)$ with $f(c) = N$.

In particular, if $f(a)f(b) < 0$, there is a $c \in (a, b)$ with $f(c) = 0$. This zero-existence statement is often called **Bolzano's theorem**. To locate a root more closely, test values inside the interval and retain a smaller interval whose endpoint values have opposite signs. This proves existence, but does not by itself prove uniqueness.

Example 1. Locate a root without solving the equation

Show that $x^3 + x - 1 = 0$ has a root between 0.6 and 0.7.

Solution. Put $f(x) = x^3 + x - 1$. It is continuous because it is a polynomial. We calculate

$$f(0.6) = 0.216 + 0.6 - 1 = -0.184, \qquad f(0.7) = 0.343 + 0.7 - 1 = 0.043.$$

The signs are opposite, so a root lies in $(0.6, 0.7)$. There is only one real root: as x increases, both x^3 and x strictly increase, so f strictly increases and cannot be zero twice.

Example 2. An intermediate value need not be zero

Show that $f(x) = x^2 + 1$ takes the value 3 on $[0, 2]$.

Solution. The function is continuous, $f(0) = 1$, and $f(2) = 5$. Since $1 < 3 < 5$, the theorem guarantees a $c \in (0, 2)$ with $f(c) = 3$. Here we can also find it: $c^2 + 1 = 3$ gives $c = \sqrt{2}$, which lies in the required interval.

Example 3. Why continuity must be checked

Let $s(x) = -1$ for $x < 0$, and $s(x) = 1$ for $x \geq 0$. Must it have a zero on $[-1, 1]$?

Solution. Its endpoint values have opposite signs, but its only output values are -1 and 1. It has no zero. The theorem does not apply because s is discontinuous at 0.

Try these

1. Use Bolzano's theorem to locate a root of $x^3 + 2x - 2 = 0$ in $(0.7, 0.8)$. Explain why there is only one real root.
2. Use the intermediate-value theorem to show that $x^2 + x = 4$ has a solution in $(0, 2)$. Then solve the quadratic and identify that solution.
3. Define $s(x) = -2$ for $x < 1$ and $s(x) = 2$ for $x \geq 1$. Does s have a zero in $[0, 2]$? Explain why its endpoint signs do not settle the question.

II.4 Inverse functions: reverse the input and output

One-to-one functions and inverse branches

A function is **one-to-one** if different inputs have different outputs. For such a function $f : D \to f(D)$, its inverse $f^{-1} : f(D) \to D$ is defined by

$$y = f(x) \quad \Longleftrightarrow \quad x = f^{-1}(y).$$

Thus the domain and range exchange roles, and $f^{-1}(f(x)) = x$ for $x \in D$. The notation f^{-1} means an inverse function; the reciprocal is $1/f$.

A strictly increasing or strictly decreasing function on an interval is one-to-one. The inverse of a continuous strictly monotone function on an interval is continuous on its range; it has the same direction of monotonicity. A function that changes direction may have several **inverse branches**, obtained by restricting its domain to separate monotone intervals. Their domains must be stated.

Example 1. Solve for the old input

Find the inverse of $f(x) = 3x - 2$, with domain $\mathbb{R}$.

Solution. Write $y = 3x - 2$ and solve: $x = (y + 2)/3$. Thus

$$f^{-1}(y) = \frac{y+2}{3}, \qquad y \in \mathbb{R}.$$

Both domain and range are $\mathbb{R}$. Substituting gives $f^{-1}(f(x)) = (3x - 2 + 2)/3 = x$, as required.

Example 2. Restrict a quadratic before inverting

Invert $f(x) = (x - 1)^2 + 2$ on the domain $[1, \infty)$.

Solution. On this domain $x - 1 \geq 0$, and the function is strictly increasing. Its range is $[2, \infty)$. From $y - 2 = (x - 1)^2$ choose the nonnegative square root:

$$f^{-1}(y) = 1 + \sqrt{y - 2}, \qquad y \geq 2.$$

On the alternative domain $(-\infty, 1]$, the inverse branch would be $1 - \sqrt{y - 2}$. On all of $\mathbb{R}$ the original function has no inverse, since, for example, $f(0) = f(2) = 3$.

Example 3. Invert a function given in pieces

Find the inverse of

$$f(x) = \begin{cases} x + 1, & x < 0, \\ 2x + 1, & x \geq 0. \end{cases}$$

Solution. The first piece has outputs $y < 1$; the second has outputs $y \geq 1$. Each piece is increasing, and the two pieces fit at the value 1. Therefore the whole function is one-to-one onto $\mathbb{R}$. Solve each rule on its own output interval:

$$f^{-1}(y) = \begin{cases} y - 1, & y < 1, \\ (y - 1)/2, & y \geq 1. \end{cases}$$

The inverse pieces also meet continuously at $y = 1$.

Try these

1. Find the inverse of $f(x) = 5 - 2x$ on $\mathbb{R}$, and check one composition.

2. Find the inverse of $f(x) = x^2$ on $[0, \infty)$. State its domain and range. Why is the same rule not invertible on all of $\mathbb{R}$?

3. Find the inverse of $f(x) = 2x$ for $x < 1$ and $f(x) = x + 1$ for $x \geq 1$. State the output interval belonging to each piece before solving.

II.5 Extrema, uniform continuity, and integral averages

Three consequences of continuity on a closed interval

If f is continuous on a closed bounded interval $[a, b]$, then:

(i) **Extreme-value theorem:** f attains a smallest and a largest value on the interval.

(ii) **Uniform continuity:** for every $\varepsilon > 0$ there is one $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ and $|x - y| < \delta$. The same δ works throughout the interval. In ordinary continuity it may depend on the point. This is the small-span idea: sufficiently short input intervals give uniformly small output variation.

(iii) **Integrability:** f is Riemann integrable on $[a, b]$. To see the connection, choose a partition whose pieces are short enough that the variation on each is less than ε. Then $U(f, P) - L(f, P) \leq \varepsilon(b - a)$, which can be made arbitrarily small.

If $a < b$, the **mean-value theorem for integrals** gives a $c \in [a, b]$ with $f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$, the **average value** of f. Indeed, comparison of integrals puts the average between the attained minimum and maximum; the intermediate-value theorem then supplies c.

Example 1. Find extrema by completing the square

Find the minimum and maximum of $f(x) = x^2 - 2x + 3$ on $[0, 3]$.

Solution. Rewrite $f(x) = (x - 1)^2 + 2$. On $[0, 3]$ the square is smallest at $x = 1$, where it is 0. Its largest value is 4 at $x = 3$; the other endpoint gives only 1. Hence the minimum is 2 at 1, and the maximum is 6 at 3.

Example 2. Choose one input allowance for the whole interval

Show directly that $f(x) = x^2$ is uniformly continuous on $[0, 2]$.

Solution. For any $x, y \in [0, 2]$, factor to obtain $|x^2 - y^2| = |x - y|\,|x + y| \leq 4\,|x - y|$. Given $\varepsilon > 0$, choose $\delta = \varepsilon/4$. Then $|x - y| < \delta$ implies $|x^2 - y^2| < \varepsilon$. This choice works throughout $[0, 2]$.

Example 3. Find the average and a point taking that value

Find the average value of $f(x) = x^2$ on $[0, 3]$, and a corresponding c.

Solution. The polynomial integral formula from Module I gives the average $\frac{1}{3} \int_0^3 x^2\, dx = \frac{1}{3} \cdot \frac{27}{3} = 3$. Since f is continuous, the theorem applies. Solve $c^2 = 3$ in $[0, 3]$: $c = \sqrt{3}$. The average height is 3.

Try these

1. Find the minimum and maximum of $x^2 - 4x + 5$ on $[0, 3]$, and all points where they occur.

2. Show directly that $f(x) = 3x - 1$ is uniformly continuous on $\mathbb{R}$ by giving a choice of δ in terms of ε.

3. Find the average value of $f(x) = 2x + 1$ on $[0, 2]$ and find a $c \in [0, 2]$ taking that value. State why the mean-value theorem applies.

Module III Differential calculus

From a difference quotient to rates of change, derivative rules, and the shape of a graph.

III.1 The derivative: slope and velocity

Definition and interpretation

The **average rate of change** of f from a to $a+h$, where $h \neq 0$, is

$$\frac{f(a+h)-f(a)}{h}.$$

The **derivative** at a is the finite limit

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h},$$

if it exists. It is the slope of the tangent line $y-f(a)=f'(a)(x-a)$. If $s(t)$ is position, then $s'(t)$ is instantaneous velocity. The limits from $h<0$ and $h>0$ must agree.

Example 1. A derivative from first principles and a tangent

Find the derivative of $f(x)=x^2$ and the tangent at $x=1$.

Solution. Expand before taking the limit:

$$\frac{f(x+h)-f(x)}{h}=\frac{(x+h)^2-x^2}{h}=2x+h \qquad (h\neq 0).$$

Thus $f'(x)=\lim_{h\to 0}(2x+h)=2x$. At $x=1$ the point is $(1,1)$ and the slope is 2. The tangent is $y-1=2(x-1)$, or $y=2x-1$.

Example 2. Average and instantaneous velocity

A particle has position $s(t)=t^2+2t$ metres at time t seconds. Find its average velocity on $[1,2]$ and its velocity at $t=1$.

Solution. Since $s(2)=8$ and $s(1)=3$, the average velocity is $(8-3)/(2-1)=5$ metres per second. For instantaneous velocity,

$$\frac{s(t+h)-s(t)}{h}=2t+2+h, \qquad s'(t)=2t+2.$$

At $t=1$ the instantaneous velocity is 4 metres per second. Average velocity uses a time interval; instantaneous velocity uses a limit.

Example 3. A corner where the derivative fails

Is $f(x)=|x|$ differentiable at 0?

Solution. Its difference quotient is $|h|/h$. This equals 1 for $h>0$ and -1 for $h<0$. The two limits disagree, so $f'(0)$ does not exist. The graph is continuous at 0, but has a corner there.

Try these

1. Use the definition to find $f'(x)$ for $f(x)=x^2+3x$. Find the tangent at $x=1$.

2. A particle has position $s(t)=t^2+4t$ metres. Find its average velocity on $[2,3]$ and its instantaneous velocity at $t=2$ seconds.

3. Use the left and right difference quotients to decide whether $f(x)=|x-1|$ is differentiable at $x=1$.

III.2 Basic derivatives and the algebra of differentiation

A small derivative toolkit

Here c is a constant, n is a positive integer, and all trigonometric angles are in **radians**.

$f(x)$	c	x^n	$\sin x$	$\cos x$	$\tan x$	e^x	$\log x$
$f'(x)$	0	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$	e^x	$1/x$

Here $\log x$ means the natural logarithm and requires $x > 0$; the tangent rule requires $\cos x \neq 0$. Also $(1/x)' = -1/x^2$ for $x \neq 0$ and $(\sqrt{x})' = 1/(2\sqrt{x})$ for $x > 0$. More generally, $(x^r)' = rx^{r-1}$ wherever the real power is differentiable.

For differentiable functions u, v,

$$\begin{aligned}(cu+v)' &= cu' + v',\\ (uv)' &= u'v + uv',\\ \left(\frac{u}{v}\right)' &= \frac{u'v - uv'}{v^2} \qquad (v \neq 0).\end{aligned}$$

Choose the rule from the *outermost operation*: sum, product, or quotient. Differentiate first; substitute a numerical value afterwards.

Example 1. A sum of familiar functions

Differentiate $f(x) = 3x^4 - 2\sqrt{x} + 5\log x$ for $x > 0$.

Solution. Differentiate each term separately:

$$f'(x) = 3(4x^3) - 2\left(\frac{1}{2\sqrt{x}}\right) + \frac{5}{x} = 12x^3 - \frac{1}{\sqrt{x}} + \frac{5}{x}.$$

Example 2. A product involving a trigonometric function

Differentiate $g(x) = x^2 \sin x$.

Solution. With $u = x^2$ and $v = \sin x$, the product rule gives

$$g'(x) = (2x)\sin x + x^2 \cos x.$$

Both terms are needed: each factor contributes to the change in the product. In particular, $(uv)'$ is generally not $u'v'$.

Example 3. A quotient

Differentiate $q(x) = (x^2+1)/(x+1)$, where $x \neq -1$.

Solution. The numerator has derivative $2x$ and the denominator has derivative 1. Hence

$$q'(x) = \frac{2x(x+1) - (x^2+1)}{(x+1)^2} = \frac{x^2 + 2x - 1}{(x+1)^2}.$$

Keep the numerator in brackets until the subtraction is complete.

Try these

1. Differentiate $f(x) = 2x^3 + \sqrt{x} - 4\log x + e^x$ for $x > 0$.

2. Differentiate $g(x) = x\cos x$ and find $g'(0)$.

3. Differentiate $q(x) = (x^2+2)/(x-1)$ for $x \neq 1$.

III.3 The chain rule, implicit differentiation, and related rates

Recognising a function inside a function

If $y = F(u)$ and $u = g(x)$, the **chain rule** is

$$\frac{dy}{dx} = F'(g(x))g'(x) = \frac{dy}{du}\frac{du}{dx}.$$

Differentiate the outer function, keep its input, and multiply by the derivative of the input. In **implicit differentiation**, y is treated as a function of x; thus $d(y^2)/dx = 2y\,dy/dx$. For **related rates**, write an equation connecting the quantities, differentiate with respect to time, then substitute the given values.

Example 1. Applying the chain rule

Differentiate $y = (3x^2 + 1)^4$.

Solution. Put $u = 3x^2 + 1$, so $y = u^4$. Then

$$\frac{dy}{dx} = 4u^3(6x) = 24x(3x^2 + 1)^3.$$

The factor $6x$ comes from the inner function; it must not be omitted.

Example 2. A tangent to a circle

For $x^2 + y^2 = 25$, find dy/dx and the tangent at $(3, 4)$.

Solution. Differentiate both sides with respect to x:

$$2x + 2y\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = -\frac{x}{y} \quad (y \neq 0).$$

At $(3, 4)$ the slope is $-3/4$. Thus the tangent is $y - 4 = -\frac{3}{4}(x - 3)$. The formula for dy/dx cannot be used where $y = 0$; the circle has vertical tangents at those points.

Example 3. A growing circle

The radius of a circle increases at 2 centimetres per second. How fast is its area increasing when its radius is 3 centimetres?

Solution. Both A and r depend on time, and $A = \pi r^2$. Differentiating with respect to t gives

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$

Now substitute $r = 3$ and $dr/dt = 2$: $dA/dt = 12\pi$ square centimetres per second. Substituting $r = 3$ *before* differentiating would incorrectly treat the radius as fixed.

Try these

1. Differentiate $y = (2x + 3)^5$ and $z = \sin(x^2)$.

2. For $x^2 + 4y^2 = 20$, find dy/dx and the tangent at $(2, 2)$.

3. A sphere has volume $V = \frac{4}{3}\pi r^3$. Its radius increases at 1 centimetre per second. Find dV/dt when $r = 2$ centimetres.

III.4 The mean-value theorem and increasing functions

From a derivative to a statement about an interval

Mean-value theorem (MVT). If f is continuous on $[a, b]$ and differentiable on (a, b), with $a < b$, then some $c \in (a, b)$ satisfies

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, a tangent is parallel to the secant joining the endpoints. When $f(a) = f(b)$ this gives $f'(c) = 0$; this special case is **Rolle's theorem**.

For f continuous on an interval and differentiable in its interior, $f' > 0$ implies f is strictly increasing, $f' < 0$ implies strictly decreasing, and $f' = 0$ everywhere implies constant. The weak inequalities $f' \geq 0$ and $f' \leq 0$ give nondecreasing and nonincreasing functions. If $|f'| \leq M$ throughout an interval, then the MVT gives $|f(v) - f(u)| \leq M\,|v - u|$ for points u, v in that interval, provided the MVT hypotheses hold between them.

Example 1. Finding the point supplied by the MVT

Apply the MVT to $f(x) = x^2$ on $[1, 3]$.

Solution. A polynomial is continuous and differentiable, so the hypotheses hold. The secant slope is $(9 - 1)/(3 - 1) = 4$. Since $f'(c) = 2c$, the required equation is $2c = 4$, giving $c = 2$. It lies inside $(1, 3)$, as required.

Example 2. Proving a function is one-to-one

Show that $f(x) = x^3 + x$ is strictly increasing on $\mathbb{R}$.

Solution. Its derivative is $f'(x) = 3x^2 + 1 > 0$ for every x. To see why this is enough, choose any $u < v$. The MVT gives

$$f(v) - f(u) = f'(c)(v - u) > 0$$

for some $c \in (u, v)$. Thus distinct inputs in increasing order have distinct outputs in increasing order. In particular, f is one-to-one.

Example 3. A bound using slopes

For $1 \leq u < v \leq 2$, prove $v^2 - u^2 \leq 4(v - u)$.

Solution. For $f(x) = x^2$, one has $f'(x) = 2x \leq 4$ on $[1, 2]$. Apply the MVT on the smaller interval $[u, v]$:

$$v^2 - u^2 = 2c(v - u) \leq 4(v - u)$$

because $u < c < v \leq 2$ and $v - u > 0$. A bound on slope therefore gives a bound on the change in height.

Try these

1. Apply the MVT to $f(x) = x^2 + 1$ on $[0, 2]$ and find c.

2. Show that $g(x) = x^3 + 2x$ is strictly increasing on $\mathbb{R}$. Explain why $g(x) = 5$ has at most one real solution.

3. Check Rolle's theorem for $f(x) = x^2 - 4x$ on $[0, 4]$ and find a point where the tangent is horizontal.

III.5 Extrema, optimisation, and a curve sketch

A practical checklist

A **local maximum** is at least as large as nearby values; a **local minimum** is at most as large as nearby values. An **absolute** maximum or minimum is compared with every value in the stated domain. An interior point where $f' = 0$ is called a stationary point. An interior extremum of a differentiable function must be stationary.

The sign change of f' decides the type: $+ \to -$ gives a local maximum, and $- \to +$ gives a local minimum. Alternatively, at a stationary point c, $f''(c) > 0$ gives a local minimum and $f''(c) < 0$ a local maximum; $f''(c) = 0$ does not decide. For a continuous function on a closed interval, find absolute extrema by comparing the values at stationary points, points where f' fails to exist, and the endpoints.

Where $f'' > 0$, the graph is concave up; where $f'' < 0$, it is concave down. An **inflection point** is a point on the graph where concavity changes. The equation $f'' = 0$ supplies possible inflection points, but a change in concavity must be checked.

Example 1. Absolute extrema on a closed interval

Find the absolute minimum and maximum of $f(x) = x^2 - 4x + 1$ on $[0, 5]$.

Solution. Since $f'(x) = 2x - 4$, the only stationary point is $x = 2$. Also $f''(x) = 2 > 0$, so it is a local minimum. For the absolute values, compare

$$f(0) = 1, \qquad f(2) = -3, \qquad f(5) = 6.$$

The absolute minimum is -3 at $x = 2$, and the absolute maximum is 6 at $x = 5$. Checking only $f' = 0$ would miss the maximum at the endpoint.

Example 2. A rectangle with largest area

A rectangle has perimeter 20 metres. Find its largest possible area.

Solution. If one side is x, the other is $10 - x$, with $0 < x < 10$. Its area is

$$A(x) = x(10 - x), \qquad A'(x) = 10 - 2x.$$

The derivative is positive for $x < 5$ and negative for $x > 5$. Thus the area increases up to $x = 5$ and decreases afterwards, so the maximum is $A(5) = 25$ square metres. The required rectangle is a 5 by 5 metre square. The restriction $0 < x < 10$ comes from positive side lengths.

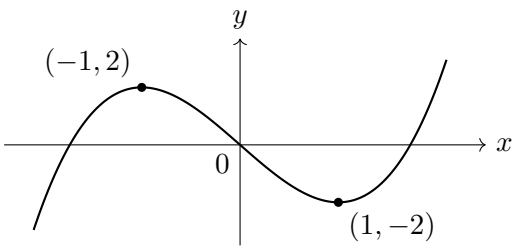
Example 3. Organising a sketch of a cubic

Describe and sketch $f(x) = x^3 - 3x$.

Solution. Its intercepts follow from $x(x^2 - 3) = 0$: $x = -\sqrt{3}, 0, \sqrt{3}$. Next,

$$f'(x) = 3(x - 1)(x + 1), \qquad f''(x) = 6x.$$

The sign of f' is positive for $x < -1$, negative for $-1 < x < 1$, and positive for $x > 1$. Hence $(-1, 2)$ is a local maximum and $(1, -2)$ is a local minimum. Since f'' changes from negative to positive at 0, the origin is an inflection point. The curve goes down without bound as $x \to -\infty$ and up without bound as $x \to \infty$.



Try these

1. Find the absolute minimum and maximum of $q(x) = x^2 - 6x + 5$ on $[0, 7]$. Check the stationary point and both endpoints.

2. A rectangle has perimeter 24 metres. Find its dimensions when the area is largest, and state that area.

3. For $g(x) = x^3 - 12x$, find the intercepts, intervals of increase and decrease, local extrema, and inflection point. Use them to sketch the curve.

Module IV Linear algebra

Start with vectors and subspaces; use bases, orthogonality and matrices to understand linear transformations.

IV.1 Vectors, linear combinations and subspaces

Definitions and a useful test

A vector in $\mathbb{R}^n$ is an ordered list of n real numbers. Add vectors and multiply by a real number component by component. A *linear combination* of $v_1, \dots, v_k$ is a vector $a_1v_1 + \cdots + a_kv_k$, where the coefficients a_i are real.

A *real linear space* (or vector space) has addition and scalar multiplication satisfying the usual arithmetic rules: addition is associative and commutative, there is a zero vector and an additive inverse for every vector, and

$$a(u+v) = au + av, \quad (a+b)v = av + bv, \quad a(bv) = (ab)v, \quad 1v = v.$$

The operations must stay inside the space. Examples are $\mathbb{R}^n$ and P_2, the space of all real polynomials of degree at most two, including the zero polynomial.

A subset W of a vector space is a *subspace* if it is itself a vector space with the same operations. To check this, it is enough to verify: $0 \in W$; $u + v \in W$ whenever $u, v \in W$; and $cu \in W$ for every real c and $u \in W$.

Example 1. Calculate a linear combination

Let $u = (1, -2)$ and $v = (3, 1)$. Find $2u - v$.

Solution. First multiply, then subtract component by component:

$$2u - v = (2, -4) - (3, 1) = (-1, -5).$$

The answer is again a vector in $\mathbb{R}^2$.

Example 2. A line through the origin and a line away from it

Is $W = \{(x, y) : y = 2x\}$ a subspace of $\mathbb{R}^2$? What about $V = \{(x, y) : y = 2x + 1\}$?

Solution. Every element of W is $(t, 2t)$. It contains $(0, 0)$; the sum of $(s, 2s)$ and $(t, 2t)$ is $(s + t, 2(s + t))$; and $c(t, 2t) = (ct, 2ct)$. Thus all three checks pass. The zero vector is not in V, since $0 \neq 2(0) + 1$. Hence V is not a subspace; one failed check is enough.

Example 3. Polynomials can be vectors

Show that $U = \{p \in P_2 : p(1) = 0\}$ is a subspace.

Solution. The zero polynomial vanishes at 1. If $p(1) = q(1) = 0$, then $(p + q)(1) = 0 + 0 = 0$ and $(cp)(1) = c\,p(1) = 0$. Thus U is a subspace. For instance, $x - 1$ and $x^2 - 1$ belong to U, and so does their sum $x^2 + x - 2$.

Try these

1. For $u = (1, 0, -1)$ and $v = (2, -1, 3)$, calculate $2u - v$ and $u + 3v$.

2. Use the subspace test on $W = \{(x, y) \in \mathbb{R}^2 : x - 2y = 0\}$. Write a general element using one real parameter.

3. Decide whether each subset of P_2 is a subspace: $U = \{p : p(0) = 0\}$ and $V = \{p : p(0) = 1\}$. Give a reason.

IV.2 Span, independence, basis and dimension

Four related ideas

The *span* of $v_1, \dots, v_k$ is the set of all their linear combinations. They are *linearly independent* if

$$a_1 v_1 + \cdots + a_k v_k = 0 \quad \Longrightarrow \quad a_1 = \cdots = a_k = 0.$$

Otherwise they are dependent. An ordered list $B = (v_1, \dots, v_k)$ is a *basis* of a space W if it is independent and spans W. Each $w \in W$ then has a unique expression $w = a_1 v_1 + \cdots + a_k v_k$; the column $(a_1, \dots, a_k)^{\mathsf{T}}$ is its *coordinate vector* $[w]_B$.

The *dimension* of a finite-dimensional space is the number of vectors in a basis. The usual basis of $\mathbb{R}^2$ is $((1,0),(0,1))$, and $(1, x, x^2)$ is a basis of P_2. Thus their dimensions are 2 and 3.

Example 1. Test a basis and find coordinates

Let $u = (1,1)$ and $v = (1,-1)$. Show that (u, v) is a basis of $\mathbb{R}^2$ and express $(4,2)$ in it.

Solution. The equation $au + bv = 0$ gives $a + b = 0$ and $a - b = 0$, so $a = b = 0$. Also, any (r, s) satisfies

$$(r,s) = \frac{r+s}{2}u + \frac{r-s}{2}v.$$

Thus the pair is independent and spans $\mathbb{R}^2$. For $(r,s) = (4,2)$, the coefficients are 3 and 1, so $[(4,2)]_{(u,v)} = (3,1)^{\mathsf{T}}$.

Example 2. Remove a redundant vector

Find a basis of the span of $u = (1,0,1)$, $v = (0,1,1)$ and $w = (2,1,3)$.

Solution. Since $w = 2u + v$, adding w does not enlarge the span. The first two coordinates of $au + bv = 0$ give $a = 0$ and $b = 0$, so u, v are independent. Therefore (u, v) is a basis of the given span, which has dimension 2.

Example 3. Compare polynomial coefficients

Show that $B = (1, 1 + x, x^2)$ is a basis of P_2 and find the coordinates of $2 + 3x + 4x^2$.

Solution. A combination has the form

$$a + b(1 + x) + cx^2 = (a + b) + bx + cx^2.$$

It is zero only when $b = c = 0$ and then $a = 0$, so the polynomials are independent. Any $r + sx + tx^2$ is obtained by choosing $b = s$, $c = t$, $a = r - s$, so they span P_2. The requested coordinates are $(-1, 3, 4)^{\mathsf{T}}$.

Try these

1. Show that $u = (1,0)$ and $v = (1,2)$ form a basis of $\mathbb{R}^2$. Find the coordinates of $(3,4)$ in this ordered basis.

2. For $u = (1,0,1)$, $v = (0,1,1)$ and $w = (1,1,2)$, find a dependence relation and a basis of their span. State its dimension.

3. For $B = (1, 1 + x, x^2)$, explain why B is a basis and find $[2 + 3x + x^2]_B$.

IV.3 Inner products, projections and Gram–Schmidt

Lengths and perpendicular directions

An *inner product* is a real-valued pairing satisfying symmetry, linearity in each argument, and $\langle u, u\rangle > 0$ for $u \neq 0$. In $\mathbb{R}^n$ the usual example is the dot product

$$u \cdot v = \sum_{i=1}^{n} u_i v_i, \qquad \|u\| = \sqrt{u \cdot u}.$$

An inner product space is also called a Euclidean space when it is finite-dimensional. Vectors are *orthogonal* if their inner product is zero. An *orthonormal* list consists of pairwise orthogonal unit vectors. For a subspace W, its *orthogonal complement* $W^{\perp}$ consists of vectors orthogonal to every element of W.

For $a \neq 0$, the projection of w onto the line spanned by a is

$$\operatorname{proj}_a w = \frac{w \cdot a}{a \cdot a} a.$$

The residual $w - \operatorname{proj}_a w$ is orthogonal to that line. More generally, for an orthonormal basis $q_1, \ldots, q_k$ of W, the projection onto W is $\sum_i (w \cdot q_i) q_i$.

For two independent vectors a, b, *Gram–Schmidt* constructs an orthonormal basis of the same span:

$$q_1 = \frac{a}{\|a\|}, \qquad v_2 = b - (b \cdot q_1) q_1, \qquad q_2 = \frac{v_2}{\|v_2\|}.$$

Subtract the projection before normalizing the second vector.

Example 1. Check orthogonality and normalize

Let $u = (1, 2)$ and $v = (2, -1)$. Find an orthonormal pair in their directions.

Solution. We have $u \cdot v = 2 - 2 = 0$ and $\|u\| = \|v\| = \sqrt{5}$. Therefore $u/\sqrt{5}$ and $v/\sqrt{5}$ are orthogonal unit vectors.

Example 2. Split a vector into parallel and perpendicular parts

Project $w = (3, 1)$ onto the line spanned by $a = (1, 1)$.

Solution. Since $w \cdot a = 4$ and $a \cdot a = 2$,

$$\operatorname{proj}_a w = 2(1, 1) = (2, 2), \qquad w - \operatorname{proj}_a w = (1, -1).$$

The residual has dot product $1 - 1 = 0$ with a, as required. Its length $\sqrt{2}$ is the distance from w to the line.

Example 3. Apply Gram–Schmidt

Orthonormalize $a = (1, 1, 0)$ and $b = (1, 0, 1)$.

Solution. First $q_1 = (1, 1, 0)/\sqrt{2}$. The component of b along a is $(b \cdot a)/(a \cdot a)\, a = a/2$. Thus

$$v_2 = (1, 0, 1) - \tfrac{1}{2}(1, 1, 0) = (\tfrac{1}{2}, -\tfrac{1}{2}, 1), \quad \|v_2\| = \sqrt{\tfrac{3}{2}}, \quad q_2 = \frac{(1, -1, 2)}{\sqrt{6}}.$$

The dot product of q_1 and q_2 is zero and each has length one. They span the same plane as a, b.

Try these

1. For $u = (1, -2, 2)$ and $v = (2, 1, 0)$, calculate $u \cdot v$, $\|u\|$ and $\|v\|$. Normalize both vectors.

2. Project $(4, 1)$ onto the line spanned by $(1, 2)$. Find the residual and check that it is perpendicular to the line.

3. Apply Gram–Schmidt, in the given order, to $(1, 0, 1)$ and $(1, 1, 1)$.

IV.4 Matrices and systems of linear equations

Operations and row reduction

An $m \times n$ matrix has m rows and n columns. Add matrices of the same size entry by entry; scalar multiplication also acts entry by entry. If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$, with

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Thus an entry of a product is a row dotted with a column. Multiplication is generally order dependent. The identity matrix I has ones on its diagonal and zeros elsewhere.

A linear system can be written $Ax = b$. Its *augmented matrix* is $[A \mid b]$. These row operations preserve its solutions: interchange two rows; multiply a row by a nonzero number; add a multiple of one row to another. Row reduction aims to create leading nonzero entries (*pivots*) and zeros below them. A row $[0 \ \cdots \ 0 \mid c]$ with $c \neq 0$ means there is no solution. If a system is consistent, a variable without a pivot can be chosen freely.

Example 1. Multiply in both orders

For $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$, compute AB and BA.

Solution. For example, the top left entry of AB is $1 \cdot 2 + 2 \cdot 1 = 4$. Computing each row–column product gives

$$AB = \begin{pmatrix} 4 & 6 \\ 1 & 3 \end{pmatrix}, \qquad BA = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}.$$

Both products exist, but they are different.

Example 2. A unique solution by elimination

Solve $x + y = 5$ and $2x - y = 1$.

Solution. Subtract twice the first row from the second:

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & -1 & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -3 & -9 \end{array}\right).$$

The second row gives $y = 3$; the first then gives $x = 2$. Substitution checks both original equations.

Example 3. Recognize no solution and infinitely many solutions

Compare the systems $x + y = 2$, $2x + 2y = 5$ and $x + y = 2$, $2x + 2y = 4$.

Solution. In each system subtract twice the first equation from the second. The first gives $0 = 1$, so it has no solution. The second gives $0 = 0$, which imposes no extra condition. Choose $x = t$ freely; then $y = 2 - t$. Its solutions are $(t, 2 - t)$ for all $t \in \mathbb{R}$.

Try these

1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. Find $A + B$, AB and BA.

2. Solve $x + 2y = 7$, $3x + 2y = 9$ by row reduction and check your answer.

3. Classify and solve, when possible: (a) $x - y = 1$, $2x - 2y = 2$; (b) $x - y = 1$, $2x - 2y = 3$.

IV.5 Linear transformations, kernel, range and rank

Definitions and the dimension formula

A map $T : V \to W$ is *linear* if $T(u + v) = T(u) + T(v)$ and $T(cu) = cT(u)$ for all vectors u, v and all real c. In particular, $T(0) = 0$. Its *kernel* (or null space) and *range* are

$$\ker T = \{v : T(v) = 0\}, \qquad \text{range}\, T = \{T(v) : v \in V\}.$$

They are subspaces. The *nullity* is $\dim(\ker T)$ and the *rank* is $\dim(\text{range}\, T)$. For finite-dimensional V,

$$\dim V = \text{nullity}\, T + \text{rank}\, T.$$

A map is *injective* (one-to-one) if equal outputs imply equal inputs; a linear map is injective exactly when its kernel is $\{0\}$. It is *surjective* (onto) if its range is the whole stated codomain. A bijective linear map is an *isomorphism*.

Example 1. Check linearity and detect a translation

Consider $T(x, y) = (x + 2y, 3y)$ and $S(x, y) = (x + 1, y)$.

Solution. For $u = (x, y)$ and $v = (r, s)$,

$$T(u + v) = (x + r + 2y + 2s, 3y + 3s) = T(u) + T(v),$$

and $T(cu) = (cx + 2cy, 3cy) = cT(u)$. Thus T is linear. But $S(0, 0) = (1, 0) \neq (0, 0)$, so S is not linear.

Example 2. Kernel and range of a map from three coordinates to two

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be $T(x, y, z) = (x + y, z)$.

Solution. To find the kernel, set $x + y = 0$ and $z = 0$. Thus

$$\ker T = \{(-t, t, 0) : t \in \mathbb{R}\} = \text{span}\{(-1, 1, 0)\}.$$

For any $(a, b) \in \mathbb{R}^2$, $T(a, 0, b) = (a, b)$. Therefore the range is $\mathbb{R}^2$ and the rank is 2. The nullity is 1, so $1 + 2 = 3$ checks the dimension formula. The map is onto but is not one-to-one, because its kernel contains nonzero vectors.

Example 3. One-to-one need not mean onto

Let $U : \mathbb{R}^2 \to \mathbb{R}^3$ be $U(x, y) = (x, y, x + y)$.

Solution. The equation $U(x, y) = 0$ forces $x = y = 0$, so U is injective. Its range is

$$\{(a, b, c) : c = a + b\} = \text{span}\{(1, 0, 1), (0, 1, 1)\}.$$

The displayed vectors are independent, so the rank is 2. This plane is not all of $\mathbb{R}^3$; for example $(0, 0, 1)$ is absent. Thus U is not onto its codomain. Here nullity $+$ rank $= 0 + 2 = 2$.

Try these

1. Decide whether $T(x, y) = (2x - y, x)$ and $S(x, y) = (x, y^2)$ are linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. For each conclusion give a calculation or counterexample.

2. For $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x - y, 2x - 2y)$, find bases of the kernel and range, rank and nullity. Is T injective? Is it onto?

3. For $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x + y, x - y)$, solve $T(x, y) = (a, b)$. Deduce its rank, nullity and whether it is an isomorphism.

IV.6 Matrix representations, composition and inverses

A matrix records the images of basis vectors

For a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, its standard matrix has columns $T(e_1), \ldots, T(e_n)$, where e_i are the standard unit vectors; then $T(x) = Ax$. For ordered bases $B = (b_1, \ldots, b_n)$ of the domain and C of the codomain, the corresponding matrix has columns $[T(b_i)]_C$ and satisfies

$$[T(v)]_C = [T]_{C \leftarrow B}[v]_B.$$

If both bases are B, write $[T]_B$. Coordinates always depend on the chosen ordered basis.

If T has standard matrix A and S has standard matrix B, then $S \circ T$, meaning "apply T first, then S", has matrix BA. A square matrix A is *invertible* if there is a matrix A^{-1} with $A^{-1}A = AA^{-1} = I$. For a 2×2 matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det A = ad - bc, \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (ad - bc \neq 0).$$

If $ad - bc = 0$, there is no inverse. When an inverse exists, $Ax = b$ has the unique solution $x = A^{-1}b$.

Example 1. Use a different basis

For $T(x, y) = (y, x)$, find its standard matrix and its matrix in $B = ((1, 1), (1, -1))$.

Solution. The standard unit vectors have images $T(1, 0) = (0, 1)$ and $T(0, 1) = (1, 0)$, so

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Write $b_1 = (1, 1)$, $b_2 = (1, -1)$. Since $T(b_1) = b_1$ and $T(b_2) = -b_2$, their B-coordinate columns are $(1, 0)^{\mathsf{T}}$ and $(0, -1)^{\mathsf{T}}$. Hence

$$[T]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For example, $(3, 1) = 2b_1 + b_2$ maps to $2b_1 - b_2 = (1, 3)$, as the formula for T predicts.

Example 2. Composition fixes the order of multiplication

Let $T(x, y) = (2x, y)$ and $S(x, y) = (x + y, y)$. Find $S \circ T$.

Solution. Their standard matrices and the product are

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus $(S \circ T)(x, y) = (2x + y, y)$. This also follows by substituting $T(x, y)$ into S. Reversing the order gives $(T \circ S)(x, y) = (2x + 2y, y)$, which is a different map.

Example 3. Find an inverse and undo a transformation

Find the inverse of $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and solve $A\binom{x}{y} = \binom{4}{7}$.

Solution. The determinant is $2 - 1 = 1$, so

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 8-7 \\ -4+7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Indeed $A\binom{1}{3} = \binom{4}{7}$. In general A sends (x, y) to $(u, v) = (x + y, x + 2y)$; subtracting the equations gives $y = v - u$ and then $x = 2u - v$, which explains the inverse formula here.

Try these

1. For $T(x, y) = (x, 2y)$, find its standard matrix and $[T]_B$ for $B = ((1, 0), (1, 1))$. Use $[T]_B$ to compute the B-coordinates of $T(3, 2)$.

2. Suppose T and S have standard matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Find the matrix of $S \circ T$ and its value at $(1, 1)$. Compare with the matrix of $T \circ S$.

3. Find the inverse of $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and use it to solve $Mv = (5, 3)^{\mathsf{T}}$. Explain why $N = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has no inverse.

Answers and short hints

Use these after making a serious attempt. Exercise numbers restart at 1 on each sheet; the sheet code below identifies the corresponding set.

I.1 Sets, Cartesian products and intervals

1. $A \times B = \{(-1,0),(-1,2),(1,0),(1,2)\}$; $B \times A = \{(0,-1),(0,1),(2,-1),(2,1)\}$.

2. $(-1,3]$; it excludes -1 and contains 0 and 3.

3. Intersection $(1,3]$; union $[-1,5)$. Check each endpoint in the original sets.

I.2 Functions, domains and graphs

1. $f(-2) = 5$, $f(0) = 1$; range $[1,5]$, since $0 \le x^2 \le 4$ on this domain.

2. $[-2,1) \cup (1,\infty)$. Require $x + 2 \ge 0$ and $x \ne 1$ simultaneously.

3. $p(0) = 0$, $p(1) = 4$, $p(2) = 5$. Draw $y = 2x$ for $x < 1$ with an open circle at $(1,2)$; draw $y = x + 3$ for $x \ge 1$ with a filled circle at $(1,4)$.

I.3 Partitions and step functions

1. Widths 1, 2, 3: subtract consecutive partition points.

2. On $(0,1),(1,2),(2,4),(4,5)$ the heights are $4, 4, -2, -2$.

3. Use $\{0,1,2,3\}$. On its three open pieces, $s + t$ has heights $3, 1, 3$, while st has heights $2, 0, 0$.

I.4 Integrating a step function

1. Integral $3(2) - 2(3) = 0$; geometric area $3(2) + 2(3) = 12$.

2. $\int_1^4 s = 4(1) + 1(2) = 6$, so $\int_1^4 (3s - 2) = 3(6) - 2(3) = 12$.

3. $2k + 1 = 9$, hence $k = 4$.

I.5 Upper sums, lower sums and integrability

1. Width $1/2$; $L_4 = \frac{1}{2}(0 + \frac{1}{2} + 1 + \frac{3}{2}) = \frac{3}{2}$ and $U_4 = \frac{1}{2}(\frac{1}{2} + 1 + \frac{3}{2} + 2) = \frac{5}{2}$.

2. Monotonicity gives $U_n - L_n = (1/n)(1^2 - 0^2) = 1/n$. Any integer $n > 20$ works, for example 21.

3. $|x| = -x$ on $[-1,0]$ is nonincreasing and $|x| = x$ on $[0,1]$ is nondecreasing. The function is piecewise monotone, hence integrable.

I.6 Polynomial integrals and basic properties

1. $2(2^3/3) + 3(2^2/2) - 2 = 28/3$; $\int_1^0 x^3\,dx = -1/4$.

2. $\int_1^4 f = 9 - 3 = 6$; the requested integral is $2(6) - (4 - 1) = 9$.

3. $6 \le \int_1^4 f \le 15$, by multiplying each bound by the interval length 3. It cannot equal 16.

I.7 Area under and between curves

1. $\int_0^2 x^2\,dx = 8/3$ square units.

2. Intersections: $(0,0)$ and $(2,4)$. The line is above the parabola on $[0,2]$ since $2x-x^2 = x(2-x) \geq 0$. Area $\int_0^2 (2x - x^2)\,dx = 4 - 8/3 = 4/3$.

3. Crossing at $x = 1$. Area $\int_0^1 (1-x)\,dx + \int_1^3 (x-1)\,dx = 1/2 + 2 = 5/2$.

II.1 Limits

1. For $x \neq 3$, cancel to get $x + 3$, so the limit is 6. Since $|(x+3) - 6| = |x - 3|$, choose $\delta = \varepsilon$.

2. Rationalising gives $1/(\sqrt{4+x} + 2)$ for $x \neq 0$; the limit is $1/4$.

3. The left limit is $1 + 1 = 2$, and the right limit is $3 - 1 = 2$. The two-sided limit exists and equals 2.

II.2 Continuity

1. Assign $f(2) = 4$, since the expression simplifies to $x + 2$ for $x \neq 2$.

2. The domain of g is $(-\infty, 3]$ and it is continuous throughout that domain. The rational function r is continuous on $\mathbb{R} \setminus \{-2\}$; it is undefined at -2.

3. Match the left limit $2k + 1$ to the value and right limit 3: $2k + 1 = 3$, so $k = 1$.

II.3 Intermediate values

1. For $f(x) = x^3 + 2x - 2$, $f(0.7) = -0.257$ and $f(0.8) = 0.112$. Continuity gives a root in $(0.7, 0.8)$. Both x^3 and $2x$ strictly increase, so their sum minus 2 is strictly increasing and has at most one root.

2. The continuous function $f(x) = x^2 + x$ has $f(0) = 0 < 4 < 6 = f(2)$. Thus a solution lies in $(0,2)$. Of the quadratic roots $(-1 \pm \sqrt{17})/2$, the required one is $(-1 + \sqrt{17})/2$.

3. There is no zero: the only values are -2 and 2. The function is discontinuous at 1, so Bolzano's theorem cannot be applied on $[0,2]$.

II.4 Inverse functions

1. $f^{-1}(y) = (5 - y)/2$, with domain and range $\mathbb{R}$. For example, $f^{-1}(f(x)) = (5 - (5 - 2x))/2 = x$.

2. $f^{-1}(y) = \sqrt{y}$, with domain $[0, \infty)$ and range $[0, \infty)$. On all of $\mathbb{R}$, f is not one-to-one: $f(1) = f(-1)$.

3. The first piece has outputs $y < 2$, and the second has outputs $y \geq 2$. Thus $f^{-1}(y) = y/2$ for $y < 2$, and $f^{-1}(y) = y - 1$ for $y \geq 2$. Its domain and range are $\mathbb{R}$.

II.5 Extrema and integral averages

1. Write $f(x) = (x-2)^2 + 1$. The minimum is 1 at $x = 2$. The maximum is 5 at $x = 0$, since on $[0,3]$ the largest distance from 2 is 2.

2. $|f(x) - f(y)| = 3\,|x - y|$, so $\delta = \varepsilon/3$ works for all real x, y.

3. The average is $\frac{1}{2}\int_0^2 (2x+1)\,dx = \frac{1}{2}(4+2) = 3$. Solve $2c + 1 = 3$ to obtain $c = 1$. The function is continuous on the closed interval $[0,2]$.

III.1 The derivative: slope and velocity

1. The difference quotient is $2x + h + 3$, so $f'(x) = 2x + 3$. At $x = 1$, $f(1) = 4$ and $f'(1) = 5$; the tangent is $y - 4 = 5(x - 1)$, or $y = 5x - 1$.

2. $s(2) = 12$, $s(3) = 21$, so average velocity is 9 metres per second. Since $s'(t) = 2t+4$, instantaneous velocity at $t = 2$ is 8 metres per second.

3. At $x = 1$, the quotient is $|h|/h$. Its left limit is -1 and its right limit is 1, so the derivative does not exist.

III.2 Basic derivatives and the algebra of differentiation

1. $f'(x) = 6x^2 + 1/(2\sqrt{x}) - 4/x + e^x$, for $x > 0$.

2. $g'(x) = \cos x - x \sin x$, so $g'(0) = 1$.

3. $q'(x) = [2x(x - 1) - (x^2 + 2)]/(x - 1)^2 = (x^2 - 2x - 2)/(x - 1)^2$.

III.3 The chain rule, implicit differentiation, and related rates

1. $dy/dx = 10(2x + 3)^4$ and $dz/dx = 2x \cos(x^2)$.

2. $2x + 8y\, dy/dx = 0$, so $dy/dx = -x/(4y)$ when $y \neq 0$. At $(2, 2)$ the slope is $-1/4$, giving $y - 2 = -\frac{1}{4}(x - 2)$.

3. $dV/dt = 4\pi r^2\, dr/dt = 16\pi$ cubic centimetres per second.

III.4 The mean-value theorem and increasing functions

1. The hypotheses hold for this polynomial. Its secant slope is $[5 - 1]/2 = 2$. Since $f'(c) = 2c = 2$, take $c = 1 \in (0, 2)$.

2. $g'(x) = 3x^2 + 2 > 0$, so g is strictly increasing by the MVT. If $u < v$ were two solutions, strict increase would give $g(u) < g(v)$, contradicting $g(u) = g(v) = 5$.

3. A polynomial meets the continuity and differentiability hypotheses, and $f(0) = f(4) = 0$. Solve $f'(c) = 2c - 4 = 0$ to obtain $c = 2 \in (0, 4)$. The horizontal tangent is at $(2, -4)$.

III.5 Extrema, optimisation, and a curve sketch

1. $q'(x) = 2x - 6$ vanishes at $x = 3$. Compare $q(0) = 5$, $q(3) = -4$, and $q(7) = 12$. The absolute minimum is -4 at $x = 3$; the absolute maximum is 12 at $x = 7$.

2. The sides are x and $12 - x$, with $0 < x < 12$. $A'(x) = 12 - 2x$ changes from positive to negative at $x = 6$. The dimensions are 6 by 6 metres and the maximum area is 36 square metres.

3. The intercepts are $(-2\sqrt{3}, 0)$, $(0, 0)$, and $(2\sqrt{3}, 0)$. Since $g' = 3(x - 2)(x + 2)$, the function increases on $(-\infty, -2)$ and $(2, \infty)$ and decreases on $(-2, 2)$. Its local maximum is $(-2, 16)$ and local minimum is $(2, -16)$. Since $g'' = 6x$, it is concave down for $x < 0$, concave up for $x > 0$, and has inflection point $(0, 0)$. The cubic tends to $-\infty$ on the left and $+\infty$ on the right.

IV.1 Vectors, linear combinations and subspaces

1. $2u - v = (0, 1, -5)$ and $u + 3v = (7, -3, 8)$.

2. Yes. A general element is $(2t, t)$. The zero vector corresponds to $t = 0$; adding two such vectors or multiplying one by a scalar gives the same form.

3. U is a subspace: evaluation at 0 is preserved under addition and scalar multiplication, and the zero polynomial belongs to U. V is not a subspace because it does not contain the zero polynomial.

IV.2 Span, independence, basis and dimension

1. $au + bv = (a + b, 2b)$. It is zero only if $a = b = 0$, and any (r, s) is obtained with $b = s/2$, $a = r - s/2$. Thus this is a basis. For $(3, 4)$, $b = 2$, $a = 1$, so the coordinate vector is $(1, 2)^{\mathsf{T}}$.

2. $u + v - w = 0$. Since u, v are independent and $w = u + v$, a basis is (u, v) and the dimension is 2.

3. Comparing coefficients in $a + b(1 + x) + cx^2$ gives unique a, b, c for every polynomial of degree at most two. For $2 + 3x + x^2$, $(a, b, c) = (-1, 3, 1)$.

IV.3 Inner products, projections and Gram–Schmidt

1. $u \cdot v = 0$, $\|u\| = 3$, $\|v\| = \sqrt{5}$. The normalized vectors are $(1, -2, 2)/3$ and $(2, 1, 0)/\sqrt{5}$.

2. The projection is $\frac{6}{5}(1, 2) = (6/5, 12/5)$. The residual is $(14/5, -7/5)$ and its dot product with $(1, 2)$ is $14/5 - 14/5 = 0$.

3. $q_1 = (1, 0, 1)/\sqrt{2}$. Subtract $\frac{(1,1,1)\cdot(1,0,1)}{2}(1, 0, 1) = (1, 0, 1)$ from the second vector, leaving $(0, 1, 0)$. Thus $q_2 = (0, 1, 0)$.

IV.4 Matrices and systems of linear equations

1. $A + B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $AB = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$, $BA = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.

2. Subtract three times the first equation from the second: $-4y = -12$, so $y = 3$ and $x = 1$. Check: $1 + 6 = 7$ and $3 + 6 = 9$.

3. (a) Infinitely many solutions $(1 + t, t)$, $t \in \mathbb{R}$: the second equation repeats the first. (b) No solution: subtracting twice the first equation gives $0 = 1$.

IV.5 Linear transformations, kernel, range and rank

1. T is linear: $T(u + v) = T(u) + T(v)$ and $T(cu) = cT(u)$ follow by expanding the coordinates. S is not: $S(0, 2) = (0, 4)$, whereas $2S(0, 1) = (0, 2)$.

2. $\ker T = \{(t, t, s) : t, s \in \mathbb{R}\}$, with basis $((1, 1, 0), (0, 0, 1))$; nullity 2. The range is $\{(r, 2r) : r \in \mathbb{R}\}$, with basis $((1, 2))$; rank 1. Since the kernel is nonzero and the range is not all of $\mathbb{R}^2$, T is neither injective nor onto. Check: $2 + 1 = 3$.

3. The coordinate formulas are linear, so T is a linear map. Adding and subtracting the output equations gives $x = (a + b)/2$ and $y = (a - b)/2$. Every output has exactly one input, so T is an isomorphism. Rank 2, nullity 0.

IV.6 Matrix representations, composition and inverses

1. The standard matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Since $T(1,0) = (1,0)$ and $T(1,1) = (1,2) = -(1,0) + 2(1,1)$,

$$[T]_B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}.$$

As $[(3,2)]_B = (1,2)^\mathsf{T}$, multiplication gives $[T(3,2)]_B = (-1,4)^\mathsf{T}$. It represents $-(1,0) + 4(1,1) = (3,4)$.

2. $[S \circ T] = BA = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, so $(S \circ T)(1,1) = (3,4)$. In the reverse order, $[T \circ S] = AB = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$.

3. $\det M = 1$ and $M^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, giving $v = (2,1)^\mathsf{T}$. For N, $\det N = 4 - 4 = 0$, so it has no inverse; its second row is twice its first.