Differential Geometry 2023 - Assignment

Answer any five out of eight. Each carries 6 marks

1. Find the length of the oriented plane curve $f^{-1}(2)$ oriented by $\nabla f/\|\nabla f\|$, where $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 - 1)^2$.

2. Let f and g be smooth functions on the open set $U \subset \mathbb{R}^{n+1}$. Show that

   (a) $d(f + g) = df + dg$

   (b) $d(fg) = gdf + fdg$

   (c) If $h : \mathbb{R} \to \mathbb{R}$ is smooth then $d(hof) = (h'of)df$

3. Let $\alpha : [a, b] \to \mathbb{R}^2 \setminus \{0\}$ be closed piecewise smooth parametrized curve. Show that the winding number of α is the same as the winding number of $f\alpha$ where $f : [a, b] \to \mathbb{R}$ is any piecewise smooth function along α with $f(a) = f(b)$ and $f(t) > 0$ for all $t \in [a, b]$. Conclude that α and $\alpha/\|\alpha\|$ have the same winding number.

4. Compute the integral $\int_C(-x_2 dx_1 + x_1 dx_2)$ where C is the ellipse $(x_1^2/a^2) + (x_2^2/b^2) = 1$ oriented by its inward normal.

5. Let S be an oriented 2-surface in $\mathbb{R}^3$ and let $p \in S$. Show that, for each $v, w \in S_p$, $L_p(v) \times L_p(w) = K(p)v \times w$.

6. Find the Gaussian curvature of $(x_1^2/a^2) + (x_2^2/b^2) - x_3 = 0$.

7. Let S be an oriented 2-surface in $\mathbb{R}^3$ and let $\{v_1, v_2\}$ be an orthonormal basis for S_p consisting of eigenvectors of L_p. Let $k_i = k(v_i)$. Show that the mean curvature at p is given by the formula

$$H(p) = \frac{1}{2\pi}\int_0^{2\pi} k(v(\theta))d\theta.$$

8. Let $S = f^{-1}(c)$ be an oriented $n-$surface, oriented by $N = \nabla f/\|\nabla\|$. Show that $H(p) = -(1/n)div\ N$.