Differential Geometry 2023 - Minor 2

PART A: each question carries a maximum of 3 marks.

1. Define the following:
   (a) Normal vector field on an n-surface
   (b) Connected subset of $\mathbb{R}^{n+1}$
   (c) Gauss map
2. Define the velocity and speed of a parametrized curve α. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.
3. Explain the positive θ-rotation at a point on an oriented 2-surface.
4. Show that the spherical image of an n-surface with orientation N is the reflection through the origin of the spherical image of the same n-surface with orientation -N.
5. Let S_1 and S_2 be two distinct n-surfaces exhibited as level sets of a smooth function $f : U \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^{n+1}$. Then $S_1 \cap S_2 = \phi$ and U can be expressed a disjoint union of level sets of f.

PART B: Each question carries a maximum of 5 marks.

1. State true or false with explanation: Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2 \text{ and } x \leq 0\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2, x > 0 \text{ and } y < 0\}$ is a plane curve in $\mathbb{R}^2$.
2. State true or false with explanation: $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + 2x = 0\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 4x - 4y + 7 = 0\}$ has exactly two smooth unit normal vector fields.
3. Let $U = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 3\}$, $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and let $f : U \to \mathbb{R}$ be a smooth function. Find g such that S is a level set of g. Then there exist a point $p \in S$ such that $f(p) \geq f(x)$ for all $x \in S$ and $\nabla f(p)$ is a scalar multiple of $\nabla g(p)$.