Ordinary Differential Equations: Notes and Problems

Based on Simmons' Differential Equations

2025 Batch

1 Power Series Solutions

1.1 Introduction: Power Series and Special Functions

Second-order linear differential equations are often solved by finding a solution in the form of a **power series**. This approach is essential for equations whose solutions cannot be expressed in terms of **elementary functions** (algebraic, trigonometric, exponential, etc.). The resulting solutions are often new functions, called **special functions** (e.g., Gamma, Riemann Zeta, Elliptic functions).

1.2 Definition of a Power Series

1. **Centered at $x = 0$:**
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad (*)$$
2. **Centered at $x = x_0$:**
$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$
   This is called a **power series in $\boldsymbol{x - x_0}$**. The discussion can usually be simplified to form (*) via a coordinate translation ($X = x - x_0$).

A series is said to **converge** at a point x if the limit of the partial sums exists: $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists. The value of this limit is the **sum** of the series.

1.3 Convergence and the Radius of Convergence (R)

A power series is characterized by its Radius of Convergence, R, where $0 \le R \le \infty$.

- **Convergence:** The series converges for all x such that $\boldsymbol{|x| < R}$ (i.e., in the interval $-R < x < R$).
- **Divergence:** The series diverges for all x such that $\boldsymbol{|x| > R}$.
- **Endpoints ($x = \pm R$):** Behavior must be checked separately.

The interval of convergence is $(-R, R)$ plus any included endpoints.

1.3.1 Finding the Radius of Convergence

The value of R can often be found using the **Ratio Test**:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

If this limit exists, it determines the radius R. If the limit is ∞, then $R = \infty$ and the series converges for all x.

1.4 Algebraic and Calculus Properties

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ both converge for $|x| < R$.

1.5 Continuity and Differentiability

If (1) converges for $|x| < R$, the function $f(x)$ defined by its sum is **automatically continuous** and has **derivatives of all orders** for $|x| < R$.

Termwise Differentiation

The series can be differentiated term by term, and the resulting series has the same radius of convergence, $\boldsymbol{R}$:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

Termwise Integration

The series can also be integrated term by term, and the resulting series also converges for $|x| < R$.

1.5.1 Algebraic Operations

1. **Addition/Subtraction:**

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

2. **Multiplication:** (If one of the functions is a polynomial or has a power series representation)

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n$$

1.6 Taylor Series and Analytic Functions

If a function $f(x)$ is represented by a power series, the coefficients a_n are uniquely determined by the derivatives of the function at the center x_0.

1.6.1 Taylor Series Formula

The coefficients of a power series expansion of $f(x)$ about x_0 are given by:

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

This gives the **Taylor series** of $f(x)$ at x_0:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

1.6.2 Analytic Functions

A function $f(x)$ is called **analytic at $\boldsymbol{x_0}$** if it can be represented by its Taylor series in some neighborhood of x_0.

Examples of Analytic Functions (at $x = 0$):

- **Exponential function:**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (R = \infty)$$

- **Sine function:**

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad (R = \infty)$$

- **Cosine function:**

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (R = \infty)$$

Polynomials, e^x, $\sin x$, and $\cos x$ are analytic at all points.

1.6.3 Key Facts about Analyticity

1. If $f(x)$ and $g(x)$ are analytic at x_0, then $f(x) \pm g(x)$, $f(x)g(x)$, and $f(x)/g(x)$ (if $g(x_0) \neq 0$) are also analytic at x_0.
2. The sum of a power series is **analytic at all points inside the interval of convergence**.