Ordinary Differential Equations 2023 - Minor 1

Write any two of the following. Each question carries 5 marks.

1. (a) State Picard's theorem. (1 mark)

   (b) State a theorem which guarantees the existence and uniqueness of the solution of the initial value problem

   $$y' = f(x,y),\ \ y(x_0) = y_0, a \leq x_0 \leq b$$

   on $[a,b]$. (1 mark)

   (c) Explain about the uniqueness of solution of

   $$y' = 3y^{\frac{2}{3}},\ \ y(0) = 0$$

   in a neighborhood of $x_0 = 0$. (3 marks)

2. (a) Write the normal form of the equation $x^2y'' + xy' + (x^2 - p^2)y = 0$ (1 mark)

   (b) Prove or disprove: If y is a non trivial solution of $y'' - x^2y - y = 0$, then y has at least two zeroes on the positive x axis. (1.5 mark)

   (c) The initial value problem of the form

   $$\begin{cases} \frac{dy}{dx} = p_1(x)y + q_1(x)z + r_1(x) & y(x_0) = y_0 \\ \frac{dz}{dx} = p_2(x)y + q_2(x)z + r_2(x) & z(x_0) = z_0 \end{cases}$$

   where the coefficient functions are continuous on an interval $[a,b]$ and $a \leq x_0 \leq b$. Sketch the proof of existence and solution on $[a,b]$.(2.5 marks)

3. (a) State Sturm comparison theorem (1 mark)

   (b) State Sturm separation theorem (1 mark)

   (c) Sketch the proof of one of the above theorems (3 marks)

Answer as much as possible (with proper justifications). Each question carries 5 marks. The maximum marks can be obtained from this section is 10.

1. Study the initial value problem $y' = sin\ y$, $y(0) = 1$ without solving the problem. Sketch the solution in the (x,y) plane.

2. Let $f : \mathbb{R} \to \mathbb{R}$ such that there exist $|f(x) - f(y)| \leq K|x-y|^\alpha$ for some $\alpha > 0$ and some $K > 0$. Prove or disprove the following

   (a) f is continuous

   (b) f is uniformly continuous

3. Let $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions (ie. linearly independent solutions) to the differential equation

$$y'' + P(x)y' + Q(x)y = 0, \ \ a \leq x \leq b$$

where P and Q are continuous functions on $[a, b]$ and x_0 is a point in (a, b). Prove or disprove the following:

   (a) Both $y_1(x)$ and $y_2(x)$ cannot have a local maximum at x_0.

   (b) Both $y_1(x)$ and $y_2(x)$ cannot have a local minimum at x_0.

   (c) $y_1(x)$ cannot have a local maximum at x_0 and $y_2(x)$ cannot have a local minimum at x_0 simultaneously.

   (d) Both $y_1(x)$ and $y_2(x)$ cannot vanish at x_0 simultaneously.