Partial Differential Equations

Course Code: MSMAT03DSC12

Classification of Second Order PDE

1. Standard Form of a Second-Order PDE

A general linear or semi-linear second-order partial differential equation in two independent variables (x, y) is written in the standard form:

$$R\,u_{xx} + S\,u_{xy} + T\,u_{yy} + f(x, y, u, u_x, u_y) = 0$$

Where:

- R, S, and T are the coefficients of the highest-order derivatives (u_{xx}, u_{xy}, u_{yy} respectively).
- These coefficients can be constants or functions of x and y.
- $f(x, y, u, u_x, u_y)$ contains all the lower-order terms (first derivatives and the function itself).

2. Classification Using R, S, T

The classification of the PDE is determined entirely by the discriminant of the principal part, calculated as $\Delta = S^2 - 4RT$.

- **Hyperbolic:** If $S^2 - 4RT > 0$. (Models wave-like propagation).
- **Parabolic:** If $S^2 - 4RT = 0$. (Models diffusion or heat flow).
- **Elliptic:** If $S^2 - 4RT < 0$. (Models steady-state or equilibrium systems).

3. Step-by-Step Procedure to Reduce to Canonical Form

Step 1: Identify Coefficients and Classify
Extract R, S, and T from the PDE and evaluate $S^2 - 4RT$ to determine the type.

Step 2: Find the Characteristic Roots
Form the characteristic quadratic equation:

$$R\lambda^2 + S\lambda + T = 0$$

Solve for λ to find the roots λ_1 and λ_2.

- Hyperbolic: Two distinct real roots.
- Parabolic: One repeated real root ($\lambda_1 = \lambda_2$).
- Elliptic: Two complex conjugate roots ($\lambda = m \pm in$).

Step 3: Solve the Characteristic ODEs
Set up the ordinary differential equations:

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0$$

Integrate both to find the characteristic curves $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$.

Step 4: Define the Canonical Variables (ξ, η)

- **Hyperbolic:** Set $\xi = f_1(x, y)$ and $\eta = f_2(x, y)$.
- **Parabolic:** Set $\xi = f_1(x, y)$ and choose η as any independent function of x and y (often simply $\eta = x$ or $\eta = y$).
- **Elliptic:** The roots give complex functions ξ and η. Transform them into real variables by taking the real and imaginary parts: $\alpha = \frac{\xi+\eta}{2}$ and $\beta = \frac{\xi-\eta}{2i}$.

Step 5: Apply the Chain Rule
Calculate the required partial derivatives in terms of the new variables:

- $u_x = u_\xi \xi_x + u_\eta \eta_x$
- $u_y = u_\xi \xi_y + u_\eta \eta_y$

Differentiate again to find u_{xx}, u_{xy}, and u_{yy}.

Step 6: Substitute and Simplify
Plug these new derivative expressions back into the original PDE. The equation will collapse into its simplified canonical form. Replace any remaining x and y variables with their equivalents in ξ and η.

4. Worked Examples

Example 1: Hyperbolic PDE

Equation: $u_{xx} - x^2 u_{yy} = 0 \quad (\text{for } x \neq 0)$

1. **Classify:** $R = 1, S = 0, T = -x^2$. Discriminant: $0 - 4(1)(-x^2) = 4x^2 > 0 \implies$ Hyperbolic.
2. **Roots:** $\lambda^2 - x^2 = 0 \implies \lambda_1 = x, \lambda_2 = -x$.
3. **ODEs:**
$$\frac{dy}{dx} + x = 0 \implies y + \frac{x^2}{2} = c_1$$
$$\frac{dy}{dx} - x = 0 \implies y - \frac{x^2}{2} = c_2$$
4. **Variables:** $\xi = y + \frac{x^2}{2}$, $\eta = y - \frac{x^2}{2}$.
5. **Derivatives:**
   - $\xi_x = x, \xi_y = 1, \eta_x = -x, \eta_y = 1$
   - $u_{xx} = x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} + u_\xi - u_\eta$
   - $u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$
6. **Substitute:**
$$(x^2 u_{\xi\xi} - 2x^2 u_{\xi\eta} + x^2 u_{\eta\eta} + u_\xi - u_\eta) - x^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 0$$
   Simplifies to: $-4x^2 u_{\xi\eta} + u_\xi - u_\eta = 0$. Since $x^2 = \xi - \eta$, the canonical form is:
$$u_{\xi\eta} = \frac{1}{4(\xi - \eta)}(u_\xi - u_\eta)$$

Example 2: Parabolic PDE

Equation: $y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = 0$

1. **Classify:** $R = y^2, S = -2xy, T = x^2$. Discriminant: $(-2xy)^2 - 4(y^2)(x^2) = 4x^2y^2 - 4x^2y^2 = 0 \implies$ Parabolic.
2. **Roots:** $y^2\lambda^2 - 2xy\lambda + x^2 = 0 \implies (y\lambda - x)^2 = 0 \implies \lambda = \frac{x}{y}$.
3. **ODE:** $\frac{dy}{dx} + \frac{x}{y} = 0 \implies y\,dy + x\,dx = 0 \implies \frac{y^2}{2} + \frac{x^2}{2} = c \implies x^2 + y^2 = c_1$.
4. **Variables:** $\xi = x^2 + y^2$. Choose $\eta = x^2 - y^2$ (or simply $\eta = x$) as an independent variable.
5. **Derivatives (using $\eta = x^2 - y^2$):**
   - $\xi_x = 2x, \xi_y = 2y, \eta_x = 2x, \eta_y = -2y$

   By applying the chain rule for u_{xx}, u_{xy}, u_{yy} and substituting, all terms involving $u_{\xi\xi}$ and $u_{\xi\eta}$ will cancel out perfectly because $\Delta = 0$.
6. **Substitute:** The equation reduces to a form containing only $u_{\eta\eta}$.

Example 3: Elliptic PDE

Equation: $u_{xx} + x^2 u_{yy} = 0$

1. **Classify:** $R = 1, S = 0, T = x^2$. Discriminant: $0 - 4(1)(x^2) = -4x^2 < 0 \implies$ Elliptic.
2. **Roots:** $\lambda^2 + x^2 = 0 \implies \lambda_1 = ix, \lambda_2 = -ix$.
3. **ODEs:**
$$\frac{dy}{dx} + ix = 0 \implies iy + \frac{x^2}{2} = c_1$$
$$\frac{dy}{dx} - ix = 0 \implies -iy + \frac{x^2}{2} = c_2$$
4. **Variables:** Complex variables are $\xi = \frac{x^2}{2} + iy$ and $\eta = \frac{x^2}{2} - iy$. Real variables: $\alpha = \mathrm{Re}(\xi) = \frac{x^2}{2}$, $\beta = \mathrm{Im}(\xi) = y$.
5. **Derivatives:**
   - $u_{xx} = u_\alpha + x^2 u_{\alpha\alpha}$
   - $u_{yy} = u_{\beta\beta}$
6. **Substitute:**
$$(u_\alpha + x^2 u_{\alpha\alpha}) + x^2(u_{\beta\beta}) = 0 \implies x^2(u_{\alpha\alpha} + u_{\beta\beta}) + u_\alpha = 0$$
   Since $x^2 = 2\alpha$, the canonical form is:
$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\alpha} u_\alpha$$

Exercise Sheet: Canonical Form Reduction

Section A: Hyperbolic PDEs

Reduce the following to canonical form by finding ξ and η.

1. $u_{xx} - 4u_{yy} = 0$
2. $u_{xx} - 5u_{xy} + 6u_{yy} = 0$
3. $y^2 u_{xx} - x^2 u_{yy} = 0 \quad (x > 0, y > 0)$
4. $u_{xx} - u_{xy} - 2u_{yy} = 0$
5. $x^2 u_{xx} - y^2 u_{yy} = 0 \quad (x > 0, y > 0)$

Section B: Parabolic PDEs

Verify these are parabolic, choose an appropriate ξ and η, and reduce.

1. $u_{xx} - 2u_{xy} + u_{yy} = 0$
2. $u_{xx} + 4u_{xy} + 4u_{yy} = 0$
3. $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0$
4. $u_{xx} - 6u_{xy} + 9u_{yy} = 0$
5. $y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} = xu_x + yu_y$

Section C: Elliptic PDEs

Identify the complex roots, separate into real coordinates α and β, and reduce.

1. $u_{xx} + 4u_{yy} = 0$
2. $u_{xx} + 2u_{xy} + 5u_{yy} = 0$
3. $x^2 u_{xx} + y^2 u_{yy} = 0 \quad (x > 0, y > 0)$
4. $yu_{xx} + u_{yy} = 0 \quad (y > 0)$
5. $u_{xx} - 2u_{xy} + 2u_{yy} = 0$