**Wave Equation- Model Questions**

1. Let $u(x,t)$ satisfy the wave equation

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \quad x \in (0, 2\pi), t > 0 \\ &u(x,0) = e^{i\omega x} \end{aligned}$$

for some $\omega \in \mathbb{R}$. Then

(a) $u(x,t) = e^{i\omega x}\ e^{i\omega t}$

(b) $u(x,t) = e^{i\omega x}\ e^{-i\omega t}$

(c) $u(x,t) = e^{i\omega x}\ \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$

(d) $u(x,t) = e^{i\omega x}\ \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$

**Solutions: a,b,c,d**

2. Let $u$ be the solution of the wave equation

$$\begin{cases} u_{tt} - u_{xx} &= 0 \qquad (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) &= f(x) \quad x \in \mathbb{R} \\ u_t(x,0) &= g(x) \quad x \in \mathbb{R} \end{cases}$$

Suppose that $f(x) = g(x) = 0$ for all $x \notin [0,1]$. Then,

(a) $u(x,t) = 0$ for all $(x,t) \in (-\infty, 0) \times (0, \infty)$

(b) $u(x,t) = 0$ for all $(x,t) \in (1, \infty) \times (0, \infty)$

(c) $u(x,t) = 0$ for all $(x,t)$ such that $x + t < 0$

(d) $u(x,t) = 0$ for all $(x,t)$ such that $x - t > 1$

**Solutions: c,d**

3. Consider the wave equation

$$\begin{cases} u_{tt} - u_{xx} &= 0 \qquad (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) &= f(x) \quad x \in \mathbb{R} \\ u_t(x,0) &= g(x) \quad x \in \mathbb{R} \end{cases}$$

Let $u_i$ be the solution of the above problem with $f_i$ and $g_i$ for $1, 2$, where $f_i, g_i : \mathbb{R} \to \mathbb{R}$ are given $C^2$ functions satisfying $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, for every $x \in [-1, 1]$. Which of the following statements are necessarily true ?

(a) $u_1(0,1) = u_2(0,1)$

(b) $u_1(1,1) = u_2(1,1)$

(c) $u_1(\frac{1}{2}, \frac{1}{2}) = u_2(\frac{1}{2}, \frac{1}{2})$

(d) $u_1(0,2) = u_2(0,2)$

**Solutions: a,c**

4. Let $u$ be the solution of the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = x|x|,\ u_t(x,0) = |x|,\ x \in \mathbb{R}$$

in the upper-half plane $\{(x,t) : x \in \mathbb{R},\ t > 0\}$, excluding the lines $x = \pm ct$. Let $t > 0$ . Then the following limits :

$$\lim_{\substack{(y,s)\to(-ct,t)\\ y<-ct}} u_{xx}(y,s),\quad \lim_{\substack{(y,s)\to(-ct,t)\\ y>-ct}} u_{xx}(y,s)$$

are, respectively

(a) $-2,\ c$
(b) $-2,\ c^2$
(c) $-2,\ 1/c$
(d) $-2,\ 1/c^2$

**Solutions: c**

5. Let $u$ be the solution of the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = x|x|,\ u_t(x,0) = |x|,\ x \in \mathbb{R}$$

in the upper-half plane $\{(x,t) : x \in \mathbb{R},\ t > 0\}$, excluding the lines $x = \pm ct$. Let $t > 0$ . Then the following limits :

$$\lim_{\substack{(y,s)\to(ct,t)\\ y<ct}} u_{xx}(y,s),\quad \lim_{\substack{(y,s)\to(ct,t)\\ y>ct}} u_{xx}(y,s)$$

are, respectively

(a) $c,\ 2$
(b) $c^2,\ 2$
(c) $1/c,\ 2$
(d) $1/c^2,\ 2$

**Solutions: c**

6. Consider the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \in \mathbb{R},$$

where $u_0$ is a $C^2$ function and $u_1$ is a $C^1$ function satisfying

$$\lim_{|x|\to\infty} \frac{u_0(x)}{|x|^\alpha} = a_0 \text{ and } \lim_{|x|\to\infty} \frac{u_1(x)}{|x|^{\alpha-1}} = b_0$$

for some $\alpha > 0$. Then the limit $\lim\limits_{t\to\infty} \frac{u(x,t)}{t^\alpha}$ equals

(a) $c^{\alpha-1}(ca_0 + \alpha b_0)$

(b) $c^{\alpha}\left(a_0 + \frac{b_0}{\alpha}\right)$

(c) $c^{\alpha}(ca_0 + \alpha b_0)$

(d) $c^{\alpha-1}\left(ca_0 + \frac{b_0}{\alpha}\right)$

**Solutions: d**

7. If $u$ is the solution of the IVP

$$\begin{aligned} &u_{tt} - u_{xx} = 6,\ x \in \mathbb{R},\ t > 0, \\ &u(x,0) = x^2,\ u_t(x,0) = 4x,\ \in \mathbb{R}, \end{aligned}$$

then $u(0,2)$ equals

(a) 16

(b) 20

(c) 8

(d) 12

**Solutions: a**

8. Let $v : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$ function such that $\varphi$ and all its derivatives up to the second order vanish outside the compact set $[a,b] \times [c,d]$ in $\mathbb{R}^2$. Put $u(x,t) = v(x + ct)$, where $c > 0$ is a constant. The value of the double integral $\iint\limits_{\mathbb{R}^2} u(x,t)(\varphi_{tt} - c^2\varphi_{xx})\, dxdt$ is

(a) 0

(b) 1

(c) Depends on $\phi$

(d) Does not depends on $\phi$

**Solutions: a,d**

9. Consider the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \\ &u(x,t) = f(x),\ \ u_t(x,0) = g(x) \end{aligned}$$

Let $A, B, C, D$ be the coordinates of a characteristic parallelogram(with $A$ and $C$ diagonally opposite). Then which of the following are possible coordinates ?

(a) $A = (1,0), B = (2,1), C = (1,2), D = (0,1)$

(b) $A = (1,0), B = (1,1), C = (1,2), D = (0,1)$

(c) $A = \left(\frac{3}{2}, \frac{1}{2}\right), B = (2,1), C = (1,2), D = \left(\frac{1}{2}, \frac{3}{2}\right)$

(d) $A = \left(\frac{3}{2}, 1\right), B = (2,1), C = (1,2), D = \left(\frac{1}{2}, \frac{3}{2}\right)$

**Solutions: a,c**

10. Let $u$ be the solution of the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \\ &u(x,t) = f(x),\ u_t(x,0) = g(x) \end{aligned}$$

such that $u(\frac{1}{2}, 0) = 1, u(0, \frac{1}{2}) = 2, u(1, \frac{1}{2}) = 3$. Then which of the following are true ?

(a) The value $u(1,1)$ can be uniquely determined and $u(1,1) = 4$
(b) The value $u(1,1)$ can be uniquely determined and $u(1,1) = 2$
(c) The value $u(\frac{1}{2}, 1)$ can be uniquely determined and $u(\frac{1}{2}, 1) = 4$
(d) The value $u(\frac{1}{2}, 1)$ can be uniquely determined and $u(\frac{1}{2}, 1) = 2$

**Solution: c**

11. Consider the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \\ &u(x,t) = f(x),\ u_t(x,0) = g(x) \end{aligned}.$$

Which of the following points belongs to the domain of dependence of $(1,1)$.

(a) (1,0)
(b) $\{(x,0) : x \in (0,1)\}$
(c) $\{(x,0) : x \in (1,2)\}$
(d) $\{(x,0) : x \in (-1,1)\}$

**Solution: a,b,c**

12. Consider the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \\ &u(x,t) = f(x),\ u_t(x,0) = g(x) \end{aligned}.$$

Which of the following points belongs to the range of influence of $(1,0)$.

(a) $\left\{(x,t) \in \mathbb{R}^2 : \sqrt{(x-1)^2 + (t-1)^2} < 1\right\}$
(b) $\left\{(x,t) \in \mathbb{R}^2 : \sqrt{(x-1)^2 + (t-1)^2} < \frac{1}{2}\right\}$
(c) $\left\{(x,t) \in \mathbb{R}^2 : \sqrt{(x-1)^2 + (t-1)^2} < \frac{1}{\sqrt{2}}\right\}$
(d) $\left\{(x,t) \in \mathbb{R}^2 : t = \frac{1}{2}, x \in \left(\frac{1}{2}, \frac{3}{2}\right)\right\}$

**Solutions: b,c,d**

13. Consider the PDE

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \\ &u(x,t) = f(x),\ u_t(x,0) = g(x) \end{aligned}$$

Let $E(t)$ be the energy function. Then which of the following are true

(a) $E(t) \leq E(0)$.
(b) $E$ is strictly increasing.
(c) $E$ is strictly decreasing.
(d) $E$ is a constant.

**Solution: a,d**