Partial Differential Equations

Course Code: MSMAT03DSC12

Problem Sheet: Preliminaries and Laplace's Equation

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Instructions. These problems check your understanding of Section 0 (Preliminaries) and Section 2 (Laplace's Equation) of the lecture notes. No proofs of theorems are asked; each problem is a short computation or a question that needs a brief reason. The bracket at the end of each problem tells you which part of the notes to use.

Notation. $B(x,r)$ is the closed ball, $B^\circ(x,r)$ the open ball, $\alpha(n)$ the volume of the unit ball in $\mathbb{R}^n$, and Φ the fundamental solution of Laplace's equation. A function is *harmonic* in U if it is C^2 and $\Delta u = 0$ in U.

Part A: Preliminaries

Problem 1. *(Volume of the unit ball)* Use $\alpha(n) = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, together with $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(s+1) = s\,\Gamma(s)$.

(a) Compute $\alpha(1)$, $\alpha(2)$, $\alpha(3)$ and $\alpha(4)$.

(b) Compute the surface area $n\alpha(n)$ of the unit sphere for $n = 1, 2, 3, 4$. Do the values for $n = 2, 3$ agree with the circumference of the unit circle and the area of the unit sphere you know?

(c) What is the "sphere" $\partial B(0,1)$ when $n = 1$? Explain why its "surface area" is 2.

[Notes: Fact 0.5]

Problem 2. *(Scaling of volume and area)*

(a) In $\mathbb{R}^3$, find the volume of $B(x,3)$ and the area of $\partial B(x,3)$ (for any center x).

(b) In $\mathbb{R}^n$, by what factor do the volume of a ball and the area of its boundary sphere change when the radius is doubled?

(c) Verify the relation $|\partial B(x,r)| = \frac{n}{r}|B(x,r)|$ for the ball in (a). For which radius r is $|\partial B(0,r)| = |B(0,r)|$ in $\mathbb{R}^n$?

[Notes: Fact 0.5]

Problem 3. *(Polar coordinates for a radial function)* Let $f(y) = |y|^2$ on $\mathbb{R}^n$ and $R > 0$.

(a) Compute $\displaystyle\int_{\partial B(0,r)} f\,dS$ and $\displaystyle\int_{B(0,R)} f\,dy$.

(b) Compute the averages $\displaystyle\fint_{\partial B(0,R)} |y|^2\,dS$ and $\displaystyle\fint_{B(0,R)} |y|^2\,dy$. Which one is larger, and why is that reasonable?

(c) Check your answer to (a) for $n = 2$ by computing $\iint_{x^2+y^2\le R^2}(x^2+y^2)\,dx\,dy$ in ordinary polar coordinates.

[Notes: Corollary 0.7(c), Definition 0.4]

Problem 4. *(Integrability of $|x|^{-m}$)* Complete the table. In each column answer *yes* or *no*.

n	m	integrable on $B(0,1)$?	integrable on $\mathbb{R}^n \setminus B(0,1)$?	locally integrable on $\mathbb{R}^n$?	integrable on $\mathbb{R}^n$?
1	$\frac{1}{2}$				
2	1				
2	2				
3	1				
3	4				
3	-1				

Then use the table to explain which of Φ, $D\Phi$, $D^2\Phi$ are locally integrable in $\mathbb{R}^3$. (Recall $|\Phi| \sim |x|^{-1}$, $|D\Phi| \sim |x|^{-2}$, $|D^2\Phi| \sim |x|^{-3}$.) [Notes: Lemma 0.10]

Problem 5. *(Locally integrable or not?)* For each function, say whether it is (i) locally integrable on the given set, and (ii) integrable on that set. Give a one-line reason.

(a) $f(x) = 1$ on $\mathbb{R}^n$.

(b) $f(x) = e^x$ on $\mathbb{R}$.

(c) $f(x) = \dfrac{1}{1+x^2}$ on $\mathbb{R}$.

(d) $f(x) = \dfrac{1}{x}$ on $(0,\infty)$, and $f(x) = \dfrac{1}{|x|}$ on $\mathbb{R}$ (with any value at 0).

(e) $f(x) = \log|x|$ on $\mathbb{R}^2$. (Use polar coordinates near 0.)

(f) $f = \chi_{[0,1]}$ (the indicator function of $[0,1]$) on $\mathbb{R}$.

[Notes: Definition 0.8, Example 0.9]

Problem 6. *(Support of a function)* Recall $\operatorname{spt} f = \overline{\{x : f(x) \neq 0\}}$. Find $\operatorname{spt} f$ in each case, and decide whether $f \in C_c(\mathbb{R})$.

(a) $f(x) = \max(0, 1 - |x|)$.

(b) $f(x) = \sin x$.

(c) $f(x) = x$.

(d) $f = \chi_{(0,1)}$.

Finally, let $g(x) = x(1-x)$ for $0 \le x \le 1$ and $g = 0$ otherwise. Find $\operatorname{spt} g$. Is g (restricted to $U = (0,1)$) in $C_c(U)$? Why or why not? [Notes: Definition 0.3(4)]

Problem 7. *(The sets U_ε and compact containment)*

(a) Describe $U_\varepsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$ for: $U = (0,1)$; $U = B^\circ(0,1) \subset \mathbb{R}^n$; the annulus $U = \{1 < |x| < 3\} \subset \mathbb{R}^n$; $U = \mathbb{R}^n$. For which ε is U_ε empty?

(b) Which of the following are true? Give a reason. (i) $(\frac{1}{4}, \frac{3}{4}) \subset\subset (0,1)$; (ii) $(0, \frac{1}{2}) \subset\subset (0,1)$; (iii) $B^\circ(0, \frac{1}{2}) \subset\subset B^\circ(0,1)$; (iv) $\{x \in \mathbb{R}^n : x_n > 1\} \subset\subset \{x \in \mathbb{R}^n : x_n > 0\}$.

(c) For $U = B^\circ(0,1) \subset \mathbb{R}^2$ and $x = (\frac{1}{2}, 0)$, find all $r > 0$ such that the closed ball $B(x,r) \subset U$.

[Notes: Definition 0.1, Lemma 0.2]

Problem 8. *(Outward normals and Green's formula on an annulus)* Let $u(y) = |y|^2$ on $\mathbb{R}^n$ and $U = \{y : \varepsilon < |y| < R\}$.

(a) Write down the outward unit normal ν of U on each of the two boundary spheres.

(b) Compute $\frac{\partial u}{\partial \nu} = \nu \cdot Du$ on $\partial B(0,R)$ and on $\partial B(0,\varepsilon)$. Explain the sign on the inner sphere.

(c) Verify Green's formula $\int_U \Delta u \, dy = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS$ by computing both sides.

[Notes: Example 0.21, Theorem 0.24(i)]

Problem 9. *(Checking the Gauss–Green theorem)* Let $U = B^\circ(0,1) \subset \mathbb{R}^2$, with outward normal $\nu = (\nu^1, \nu^2) = (\cos\theta, \sin\theta)$ at the boundary point $(\cos\theta, \sin\theta)$. Verify $\int_U u_{x_i} \, dx = \int_{\partial U} u\nu^i \, dS$ in the following cases, by computing both sides:

(a) $u = x_1$, $i = 1$;

(b) $u = x_1 x_2$, $i = 2$;

(c) $u = x_1^2 + x_2^2$, $i = 1$.

[Notes: Theorem 0.22]

Problem 10. *(The standard mollifier)* Let η be the standard mollifier and $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$.

(a) Find $\operatorname{spt}\eta_\varepsilon$, and express $\eta_\varepsilon(0)$ in terms of $\eta(0)$. What happens to $\eta_\varepsilon(0)$ as $\varepsilon \to 0$?

(b) Show that $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$ using the substitution $x = \varepsilon z$.

(c) Let $U = (0,1)$ and $\varepsilon = \frac{1}{4}$. On which set is $f^\varepsilon = \eta_\varepsilon * f$ defined? Why can it not be defined at $x = \frac{1}{10}$?

(d) If $f \equiv 1$ on U, compute $f^\varepsilon(x)$ for $x \in U_\varepsilon$.

[Notes: Definition 0.26 and the definition of mollification]

Part B: Laplace's Equation

Problem 11. *(Harmonic or not?)* Decide which functions are harmonic in the indicated region, by computing Δu.

(a) $u = x^2 - y^2$ in $\mathbb{R}^2$.

(b) $u = e^x \cos y$ in $\mathbb{R}^2$.

(c) $u = \log(x^2 + y^2)$ in $\mathbb{R}^2 \setminus \{0\}$.

(d) $u = xyz$ in $\mathbb{R}^3$.

(e) $u = x^2 + y^2$ in $\mathbb{R}^2$.

(f) Find all constants a, b, c for which $u = ax^2 + by^2 + cz^2$ is harmonic in $\mathbb{R}^3$.

[Notes: Definitions of Section 2]

Problem 12. *(Radial harmonic functions)* Let $u(x) = |x|^p$ on $\mathbb{R}^n \setminus \{0\}$, $p \in \mathbb{R}$.

(a) Use the radial formula $\Delta u = v''(r) + \frac{n-1}{r} v'(r)$, where $u(x) = v(|x|)$, to show $\Delta|x|^p = p(p+n-2)|x|^{p-2}$.

(b) For which p is $|x|^p$ harmonic in $\mathbb{R}^n \setminus \{0\}$? Compare with the formula for Φ when $n \geq 3$.

(c) What happens when $n = 2$? Which radial function replaces $|x|^{2-n}$?

[Notes: Derivation of the Fundamental Solution]

Problem 13. *(Fundamental solution and the flux)*

(a) Using $\alpha(3) = \frac{4\pi}{3}$, show that in $\mathbb{R}^3$, $\Phi(x) = \dfrac{1}{4\pi|x|}$.

(b) Compute $D\Phi(x)$ for $n = 3$ and check it against $D\Phi(y) = -\dfrac{1}{n\alpha(n)} \dfrac{y}{|y|^n}$.

(c) On $\partial B(0,\varepsilon)$ with the *outward* normal $\nu = x/\varepsilon$, compute $\frac{\partial \Phi}{\partial \nu}$ and $\int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \nu} \, dS$. Does the answer depend on ε?

(d) Repeat (c) for $n = 2$, $\Phi(x) = -\frac{1}{2\pi}\log|x|$. Why do we say that Φ is "normalised so that the flux is one"?

[Notes: Defining the Fundamental Solution, formula (7)]

Problem 14. *(A solution of Poisson's equation)*

(a) Show that $u(x) = -\dfrac{|x|^2}{2n}$ solves $-\Delta u = 1$ in $\mathbb{R}^n$.

(b) Find two more solutions of $-\Delta u = 1$ in $\mathbb{R}^n$ (add a harmonic function).

(c) Why does (b) not contradict the Uniqueness theorem (Theorem 5)?

(d) Why can we not obtain a solution of $-\Delta u = 1$ from the formula $u = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy$ with $f \equiv 1$, $n = 3$? (Is the integral finite?)

[Notes: Theorem 1, Theorem 5, Lemma 0.10]

Problem 15. *(The Dirac delta as a limit)*

(a) In $\mathbb{R}$, let $\psi_k = \frac{k}{2}\chi_{[-1/k,\,1/k]}$. Check that $\int_{\mathbb{R}} \psi_k = 1$, and compute $\int_{\mathbb{R}} \psi_k(y)e^y\,dy$ explicitly. Find its limit as $k \to \infty$.

(b) What is $\lim_{k\to\infty} \psi_k(y)$ for $y \neq 0$, and for $y = 0$? Explain why this pointwise limit is *not* a function with integral 1.

(c) In $\mathbb{R}^n$, what is $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \eta_\varepsilon(y)\cos|y|\,dy$? And $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \eta_\varepsilon(y-a)g(y)\,dy$ for continuous g and a fixed point a?

[Notes: The Dirac Delta Notation]

Problem 16. *(Verifying the mean-value formula)* Let $u(x,y) = x^2 - y^2$ and fix a point $(a,b) \in \mathbb{R}^2$ and $r > 0$.

(a) Parametrise $\partial B((a,b),r)$ by $(a{+}r\cos t,\ b{+}r\sin t)$, $0 \le t \le 2\pi$, and compute $⨍_{\partial B((a,b),r)} u\,dS$. Compare with $u(a,b)$.

(b) Do the same for $w(x,y) = x^2 + y^2$. Why does the mean-value formula fail for w?

[Notes: Theorem 2]

Problem 17. *(Using the mean-value formula)* Let u be harmonic in $B^\circ(0,2) \subset \mathbb{R}^2$, and suppose that on the unit circle

$$u(\cos\theta, \sin\theta) = 3 + 2\cos\theta - \sin 3\theta \qquad (0 \le \theta \le 2\pi).$$

(a) Why may we apply the mean-value formula on the closed ball $B(0,1)$?

(b) Find $u(0,0)$.

(c) Find $⨍_{B(0,1)} u\,dy$.

[Notes: Theorem 2 and the remark after it]

Problem 18. *(The ball must lie inside the domain)* Let $n = 3$, $U = \mathbb{R}^3 \setminus \{0\}$ and $u = \Phi = \dfrac{1}{4\pi|x|}$, which is harmonic in U. Let $x = (2,0,0)$.

(a) For which radii r does the closed ball $B(x,r)$ lie in U?

(b) It is known that $⨍_{\partial B(x,r)} \dfrac{1}{|y|}\,dS(y) = \dfrac{1}{r}$ whenever $r > |x|$. Use this to compute $⨍_{\partial B(x,3)} u\,dS$ and compare it with $u(x)$.

(c) Explain why this does not contradict Theorem 2.

[Notes: Theorem 2 and the remark after it, Lemma 0.2]

Problem 19. *(The maximum principle in examples)* Let $U = B^\circ(0,1) \subset \mathbb{R}^2$.

(a) For $u = x^2 - y^2$, find $\max_{\bar{U}} u$ and $\min_{\bar{U}} u$ and the points where they are attained. (Use the maximum principle: look only at the boundary, $x = \cos t$, $y = \sin t$.)

(b) Is $(0,0)$ a maximum or a minimum point of u? Explain using the strong maximum principle.

(c) The function $v = 1 - x^2 - y^2$ attains its maximum at the interior point $(0,0)$. Why does this not contradict Theorem 4?

[Notes: Theorem 4 and the minimum principle]

Problem 20. *(Strong maximum principle: true or false)* Say *true* or *false* and give a reason or a counterexample. In each case U is bounded and open, and $u \in C^2(U) \cap C(\bar{U})$ is harmonic in U.

(a) If $U = B^\circ(0,1)$, $u = g$ on ∂U, $g \geq 0$ and $g(1,0,\ldots,0) > 0$, then $u > 0$ everywhere in U.

(b) If U is the union of two disjoint open discs, $u = g$ on ∂U, $g \geq 0$, and g is positive somewhere, then $u > 0$ everywhere in U.

(c) If U is connected and u attains its maximum over $\bar{U}$ at a point of U, then u is constant on $\bar{U}$.

(d) If U is *not* connected and u attains its maximum over $\bar{U}$ at a point of U, then u is constant on U.

(e) If $u = 0$ on ∂U, then $u \equiv 0$ in U.

[Notes: Theorem 4, the remark on positivity, the topological remark]

Problem 21. *(Uniqueness)*

(a) Let $U = B^\circ(0,1) \subset \mathbb{R}^n$. Show that $u(x) = \dfrac{1 - |x|^2}{2n}$ solves $-\Delta u = 1$ in U, $u = 0$ on ∂U. Why is it the *only* solution in $C^2(U) \cap C(\bar{U})$?

(b) Let $U = \{x \in \mathbb{R}^n : x_n > 0\}$ (a half-space). Find two different harmonic functions in U, continuous on $\bar{U}$, which both vanish on ∂U. Which hypothesis of Theorem 5 fails?

[Notes: Theorem 5 and the remark after it]

Problem 22. *(Regularity: true or false)* Say *true* or *false* with a short reason.

(a) If $u \in C^2(U)$ is harmonic, then $u \in C^\infty(U)$.

(b) If u is harmonic in U, then $u_{x_1 x_2}$ is harmonic in U.

(c) If $u \in C(U)$ satisfies the mean-value property $u(x) = \fint_{\partial B(x,r)} u \, dS$ for every ball $B(x,r) \subset U$, then u is harmonic in U.

(d) If u is harmonic in $U = B^\circ(0,1) \subset \mathbb{R}^2$, then u is bounded on U. (Consider $u(x) = \log|x - (1,0)|$.)

(e) If u is harmonic, then so is $|u|$. (Consider $u = x_1$.)

[Notes: Theorem 6, its corollary, Theorem 3]

Problem 23. *(Derivative estimates and Liouville's theorem)*

(a) Let u be harmonic in $B^\circ(0,3) \subset \mathbb{R}^2$ with $|u| \leq 10$. Use Theorem 7 with $k = 1$ on the ball $B(0,2)$ (and $\alpha(2) = \pi$) to find a number K with $|u_{x_1}(0)| \leq K$.

(b) Is $e^x \cos y$ bounded on $\mathbb{R}^2$? Does this contradict Liouville's theorem?

(c) Let u be harmonic in $\mathbb{R}^2$ with $|u| \leq 5$. What can you say about u?

(d) Let u be harmonic in $\mathbb{R}^n$ with $u(x) \to 0$ as $|x| \to \infty$. Show that $u \equiv 0$. (First explain why u is bounded.)

[Notes: Theorem 7, Theorem 8]

Problem 24. *(The representation formula)* Let $f \in C_c^2(\mathbb{R}^3)$ and $\tilde{u}(x) = \int_{\mathbb{R}^3} \Phi(x-y) f(y)\, dy$.

(a) Compute $\Phi(1)$, the value of Φ on the unit sphere in $\mathbb{R}^3$, and explain why $0 < \Phi(y) \le \Phi(1)$ for $|y| \ge 1$.

(b) $u = \tilde{u} + x_1$ also solves $-\Delta u = f$. Does this contradict Theorem 9?

(c) For $n = 2$, what is $\lim_{|x|\to\infty} \Phi(x)$? Which step of the proof of Theorem 9 breaks down?

[Notes: Theorem 9 and the remarks]

Problem 25. *(Harnack's inequality in an example)* Let $U = B^\circ(0,4) \subset \mathbb{R}^2$ and $V = B^\circ(0,1)$.

(a) Check that $u = x_1 + 4$ is a nonnegative harmonic function in U. Find $\sup_V u$, $\inf_V u$ and their ratio.

(b) For $u_t = x_1 + t$ with $t \ge 4$, find $\sup_V u_t / \inf_V u_t$. Show that one constant C works for all these functions.

(c) Replace V by $V' = B^\circ(0, 3.9)$ and compute the ratio for $u = x_1 + 4$. What does this suggest about the constant when V comes close to ∂U?

(d) Why does Harnack's inequality say nothing about $u = x_1$?

[Notes: Theorem 11 and the remark after it]

Problem 26. *(Harmonic polynomials)*

(a) Show that $x^3 - 3xy^2$ and $3x^2y - y^3$ are harmonic in $\mathbb{R}^2$.

(b) Find all harmonic polynomials in $\mathbb{R}^2$ of the form $ax^2 + bxy + cy^2$.

(c) Is the product of two harmonic functions harmonic? Test with $u = v = x$, and with $u = x$, $v = y$.

[Notes: Definitions of Section 2]

Problem 27. *(Averages without integrating)* Use the mean-value formulas to compute the following *without* evaluating any integral. First check that the function is harmonic.

(a) $\fint_{B((1,2),1)} (x^2 - y^2 + 3x)\, dx\, dy$ in $\mathbb{R}^2$.

(b) $\fint_{\partial B(0,5)} e^x \cos y\, dS$ in $\mathbb{R}^2$.

(c) $\fint_{B((1,1,1),2)} xyz\, dx\, dy\, dz$ in $\mathbb{R}^3$.

[Notes: Theorem 2]

Problem 28. *(The function $\phi(r)$ for a non-harmonic function)* Let $u(y) = |y|^2$ in $\mathbb{R}^2$ and $\phi(r) = \fint_{\partial B(0,r)} u\, dS$.

(a) Compute $\phi(r)$ directly. (On $\partial B(0,r)$, $|y| = r$.) Compare $\lim_{r\to 0} \phi(r)$ with $u(0)$, and $\phi(r)$ with $u(0)$.

(b) In the proof of Theorem 2 we showed $\phi'(r) = \frac{r}{n} \fint_{B(0,r)} \Delta u\, dy$ for every C^2 function. Check this identity here.

(c) Using (a) and Theorem 2, explain why u cannot be harmonic, without computing Δu.

[Notes: Proof of Theorem 2, formula (15)]

Problem 29. *(A boundary-value problem)* Let $U = B^\circ(0,1) \subset \mathbb{R}^2$ and $u(x,y) = \dfrac{1 + x^2 - y^2}{2}$.

(a) Show that u is harmonic and that $u = x^2$ on ∂U. (On ∂U, $y^2 = 1 - x^2$.)

(b) Why is u the only function in $C^2(U) \cap C(\bar{U})$ with these properties?

(c) Without further computation, find $\max_{\bar{U}} u$ and $\min_{\bar{U}} u$ and say where they are attained.

(d) Find $u(0,0)$ in two ways: from the formula, and as the average of the boundary values x^2 over the unit circle.

[Notes: Theorems 2, 4, 5]

Problem 30. *(New harmonic functions from old)* Let u be harmonic in $\mathbb{R}^n$. Fix $a \in \mathbb{R}^n$, $\lambda > 0$, and an orthogonal matrix O.

(a) Show that $v_1(x) = u(x - a)$ is harmonic.

(b) Show that $\Delta v_2(x) = \lambda^2 (\Delta u)(\lambda x)$ for $v_2(x) = u(\lambda x)$, so that v_2 is harmonic.

(c) Why is $v_3(x) = u(Ox)$ harmonic?

(d) Use (a) to explain why $\log|x - (1,0)|$ is harmonic in $\mathbb{R}^2 \setminus \{(1,0)\}$.

[Notes: Rotational Invariance of the Laplacian]