Partial Differential Equations

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1 Method of Characteristics

1.1 Introduction and Linear Equations

1.1.1 Introduction

The method of characteristics is a powerful technique for solving first-order partial differential equations (PDEs). The fundamental idea is to reduce the PDE to a system of ordinary differential equations (ODEs) along specific curves known as *characteristic curves*. By solving these ODEs, we can reconstruct the solution surface $u(x, y)$.

1.1.2 Linear Equations

Consider the general linear first-order PDE in two variables:

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + d(x, y) \tag{1}$$

We wish to find a solution $u(x, y)$ that satisfies a given initial condition on a curve Γ.

Step-by-step Method:

1. **Parameterize the initial curve:** Let the initial data be given on a curve Γ, parameterized by s:
$$x = x_0(s), \quad y = y_0(s), \quad u = u_0(s)$$

2. **Form the Characteristic Equations:** We look for characteristic curves $(x(\tau, s), y(\tau, s))$ parameterized by τ, emanating from the initial curve at $\tau = 0$. The characteristic ODEs are:
$$\frac{dx}{d\tau} = a(x, y), \quad \frac{dy}{d\tau} = b(x, y)$$
Along these curves, the PDE simplifies to an ODE for $u(\tau, s)$:
$$\frac{du}{d\tau} = c(x, y)u + d(x, y)$$

3. **Solve the ODEs:** Solve the system of ODEs subject to the initial conditions at $\tau = 0$:
$$x(0, s) = x_0(s), \quad y(0, s) = y_0(s), \quad u(0, s) = u_0(s)$$

4. **Invert the Coordinates:** Solve for τ and s in terms of x and y. Substitute back to find $u(x, y)$.

1.2 Quasilinear Equations

A quasilinear equation is linear in the derivatives u_x and u_y, but the coefficients can depend on u:

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{2}$$

1.2.1 Burgers' Equation

A classic example is the inviscid Burgers' equation:

$$u_t + uu_x = 0$$

with initial data $u(x, 0) = h(x)$. The characteristic equations are:

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = 0$$

This implies that u is constant along characteristics, and the characteristics are straight lines with slope (velocity) equal to the initial value $h(x)$. The solution implicitly is $u = h(x - ut)$.

Shock and Rarefaction Waves:

- **Shock Wave:** If $h(x)$ is decreasing, the characteristics will eventually intersect. At the intersection, the solution becomes multi-valued and a singularity develops. We define a *shock wave* as a propagating discontinuity that satisfies a specific physical condition (Rankine-Hugoniot jump condition) to remain a valid weak solution.
- **Rarefaction Wave:** If $h(x)$ has a discontinuous jump where it increases, a gap opens between characteristics. This region is filled by a continuous fan of characteristics known as a *rarefaction wave*, smoothing out the initial discontinuity over time.

1.2.2 Definition of Integral Surface

An *integral surface* for the PDE is the geometric surface $z = u(x, y)$ in $\mathbb{R}^3$ that represents the solution. The PDE $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$ states that the vector field (a, b, c) is always tangent to the integral surface. Therefore, the integral surface is constructed by taking the union of all characteristic curves passing through the initial data curve.

1.2.3 Theorem and Proof (Application of Implicit Function Theorem)

Theorem (Cauchy Problem): Suppose a, b, c and the initial curve Γ are C^1. If the transversality condition holds:

$$a(x_0(s), y_0(s), u_0(s))y_0'(s) - b(x_0(s), y_0(s), u_0(s))x_0'(s) \neq 0$$

then there exists a unique C^1 solution $u(x, y)$ in a neighborhood of the initial curve.

Proof in Steps:

1. **Characteristics:** Construct the system of ODEs: $\frac{dx}{d\tau} = a$, $\frac{dy}{d\tau} = b$, $\frac{du}{d\tau} = c$ with initial data at $\tau = 0$ parameterized by s. By the Picard-Lindelöf theorem, there exist unique solutions $x(\tau, s), y(\tau, s), u(\tau, s)$.
2. **Applying the Implicit Function Theorem (IFT):** To define u as a function of x and y, we need to invert the mapping $(\tau, s) \mapsto (x, y)$. The IFT states that this mapping is locally invertible near $\tau = 0$ if and only if the Jacobian determinant is non-zero:
$$J = \det \begin{pmatrix} \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial \tau} & \frac{\partial y}{\partial s} \end{pmatrix} = \frac{\partial x}{\partial \tau}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial \tau}\frac{\partial x}{\partial s} \neq 0$$
3. **Checking the Requirement:** At $\tau = 0$, we have $\frac{\partial x}{\partial \tau} = a$, $\frac{\partial y}{\partial \tau} = b$, $\frac{\partial x}{\partial s} = x_0'(s)$, and $\frac{\partial y}{\partial s} = y_0'(s)$. Therefore, the Jacobian is exactly the transversality condition: $ay_0' - bx_0'$. Since this is non-zero by assumption, the IFT guarantees we can locally find $\tau(x, y)$ and $s(x, y)$.
4. **Conclusion:** Substituting these back, we get $u(x, y) = u(\tau(x, y), s(x, y))$, which is the unique C^1 solution.

1.3 General First-Order Equation in Two Variables

Consider the fully nonlinear equation:

$$F(x, y, u, p, q) = 0$$

where $p = u_x$ and $q = u_y$.

1.3.1 Idea of the Proof in Steps

To solve this, we cannot simply use the vector field (a, b, c) as in the quasilinear case because the coefficients now depend nonlinearly on p and q. The idea is to track not just the spatial coordinates (x, y) and value u, but also the derivatives p and q along the characteristic curve.

1. Differentiate the PDE with respect to x and y to obtain relations for p_x, p_y, q_x, q_y.
2. Construct a system of five ODEs for the five variables (x, y, u, p, q). These are called the *Characteristic Strip* equations.
3. Solve this system subject to initial conditions. Ensure the initial data for $p_0(s)$ and $q_0(s)$ are compatible with $F = 0$ and the strip condition $u_0'(s) = p_0(s)x_0'(s) + q_0(s)y_0'(s)$.

1.3.2 Step-by-Step Calculation

Differentiating $F(x, y, u, p, q) = 0$ with respect to x:

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0 \implies F_x + pF_u + p_x F_p + p_y F_q = 0$$

(using $q_x = p_y$). We choose our parameter τ such that:

$$\frac{dx}{d\tau} = F_p, \quad \frac{dy}{d\tau} = F_q$$

Then along this curve, $\frac{dp}{d\tau} = p_x \frac{dx}{d\tau} + p_y \frac{dy}{d\tau} = p_x F_p + p_y F_q = -(F_x + pF_u)$. Similarly, $\frac{dq}{d\tau} = -(F_y + qF_u)$. The change in u is $\frac{du}{d\tau} = p\frac{dx}{d\tau} + q\frac{dy}{d\tau} = pF_p + qF_q$. These form the characteristic equations.

1.3.3 Geometric Explanation of the Monge Cone

At any point (x_0, y_0, u_0), the PDE restricts the possible tangent planes $(p, q, -1)$ to the solution surface via $F(x_0, y_0, u_0, p, q) = 0$. Since $F = 0$ generally defines a 1-parameter family of planes in $\mathbb{R}^3$, the envelope of all these possible tangent planes forms a cone, called the **Monge Cone**. The integral surface representing the solution must be tangent to a Monge cone at every point. The method of characteristics traces the paths that lie exactly along the contact lines between the integral surface and these Monge cones.

1.4 First-Order Equation in Several Variables

Here, we generalize the ideas to n independent variables $x = (x_1, \dots, x_n)$.

1.4.1 Linear First-Order Equation

The equation is $\sum_{i=1}^{n} a_i(x)u_{x_i} = c(x)u + d(x)$. The characteristic equations simply extend to n spatial ODEs: $\frac{dx_i}{d\tau} = a_i(x)$, along which $\frac{du}{d\tau} = c(x)u + d(x)$. The initial data is given on an $(n-1)$-dimensional hypersurface, and the solution is reconstructed by tracing characteristics from this hypersurface.

1.4.2 Quasilinear Equation

For $\sum_{i=1}^{n} a_i(x,u)u_{x_i} = c(x,u)$, the characteristic equations are $\frac{dx_i}{d\tau} = a_i(x,u)$ and $\frac{du}{d\tau} = c(x,u)$. The geometry is identical to the 2D case, where the vector field $(a_1,\ldots,a_n,c)$ in $\mathbb{R}^{n+1}$ must be tangent to the n-dimensional integral surface. The Implicit Function Theorem is again used to verify that the characteristics do not run parallel to the initial hypersurface.

1.4.3 General Nonlinear Equation

For $F(x,u,Du) = 0$ where $Du = (p_1,\ldots,p_n)$, we form the characteristic strip in $2n+1$ dimensions (x,u,p). The equations generalize gracefully:

$$\frac{dx_i}{d\tau} = F_{p_i}, \quad \frac{du}{d\tau} = \sum_{i=1}^{n} p_i F_{p_i}, \quad \frac{dp_i}{d\tau} = -(F_{x_i} + p_i F_u)$$

The initial hypersurface must be augmented with compatible values for all partial derivatives p_i, satisfying both the PDE itself and the geometric strip condition on the hypersurface. Once again, by solving this extended system of ODEs, we can construct the unique local solution.

2 Laplace's Equation

2.1 Definitions

The **Laplace equation** is given by:

$$\Delta u = 0$$

where $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$ is the Laplacian of u. A C^2 function u satisfying Laplace's equation is called a **harmonic function**. If a source term f is present, the equation becomes **Poisson's equation**:

$$-\Delta u = f$$

2.2 Rotational Invariance of the Laplacian

The Laplacian operator is invariant under rotations of the coordinate system.

2.2.1 2D Case with 2×2 Matrix

Let $u(x,y)$ be a function in $\mathbb{R}^2$. Consider a rotation matrix:

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Let new coordinates be $(x',y')^T = O(x,y)^T$. Since $O^T O = I$, we have $(x,y)^T = O^T(x',y')^T$, meaning:

$$x = x'\cos\theta + y'\sin\theta$$

$$y = -x'\sin\theta + y'\cos\theta$$

Let $v(x',y') = u(x(x',y'),y(x',y'))$. Using the chain rule:

$$v_{x'} = u_x \frac{\partial x}{\partial x'} + u_y \frac{\partial y}{\partial x'} = u_x\cos\theta - u_y\sin\theta$$

$$v_{y'} = u_x \frac{\partial x}{\partial y'} + u_y \frac{\partial y}{\partial y'} = u_x\sin\theta + u_y\cos\theta$$

Taking second derivatives:

$$v_{x'x'} = (u_{xx}\cos\theta - u_{yx}\sin\theta)\cos\theta - (u_{xy}\cos\theta - u_{yy}\sin\theta)\sin\theta$$
$$v_{x'x'} = u_{xx}\cos^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta$$
$$v_{y'y'} = (u_{xx}\sin\theta + u_{yx}\cos\theta)\sin\theta + (u_{xy}\sin\theta + u_{yy}\cos\theta)\cos\theta$$
$$v_{y'y'} = u_{xx}\sin^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta$$

Adding them together:

$$v_{x'x'} + v_{y'y'} = u_{xx}(\cos^2\theta + \sin^2\theta) + u_{yy}(\sin^2\theta + \cos^2\theta) = u_{xx} + u_{yy}$$

Thus, $\Delta_{x'} v = \Delta_x u$.

2.2.2 $\mathbb{R}^n$ Case

Let $y = Ox$ where O is an $n\times n$ orthogonal matrix ($O^T O = I$, or $\sum_k O_{ki}O_{kj} = \delta_{ij}$). Define $v(x) = u(Ox)$. Using index notation, $y_i = \sum_j O_{ij}x_j$, so $\frac{\partial y_i}{\partial x_k} = O_{ik}$. By the chain rule:

$$v_{x_k} = \sum_i u_{y_i}\frac{\partial y_i}{\partial x_k} = \sum_i u_{y_i} O_{ik}$$
$$v_{x_k x_k} = \sum_i \sum_j u_{y_i y_j}\frac{\partial y_j}{\partial x_k} O_{ik} = \sum_{i,j} u_{y_i y_j} O_{jk} O_{ik}$$

Summing over k to form the Laplacian:

$$\Delta v = \sum_k v_{x_k x_k} = \sum_{i,j} u_{y_i y_j}\left(\sum_k O_{ik}O_{jk}\right) = \sum_{i,j} u_{y_i y_j}\delta_{ij} = \sum_i u_{y_i y_i} = \Delta u$$

This rotational invariance suggests that we should look for radial solutions to Laplace's equation.

2.3 Derivation of the Fundamental Solution

We seek a solution $u(x) = v(r)$ where $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

2.3.1 Radial Formula for Laplacian

First, observe $\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \cdots + x_n^2)^{-1/2} 2x_i = \frac{x_i}{r}$. By the chain rule:

$$u_{x_i} = v'(r)\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r}$$

Now we compute the second derivative:

$$u_{x_i x_i} = \frac{\partial}{\partial x_i}\left(v'(r)\frac{x_i}{r}\right) = v''(r)\left(\frac{x_i}{r}\right)^2 + v'(r)\left(\frac{1\cdot r - x_i(x_i/r)}{r^2}\right)$$
$$u_{x_i x_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

Summing over $i = 1\dots n$:

$$\Delta u = \sum_{i=1}^n u_{x_i x_i} = v''(r)\frac{\sum x_i^2}{r^2} + v'(r)\left(\frac{n}{r} - \frac{\sum x_i^2}{r^3}\right)$$

Since $\sum x_i^2 = r^2$, we obtain the radial Laplacian:

$$\Delta u = v''(r) + \frac{n-1}{r}v'(r) = 0$$

2.3.2 Solving the ODE

Let $w = v'$. The ODE becomes $w' + \frac{n-1}{r} w = 0$.

$$\frac{w'}{w} = \frac{1-n}{r} \implies \ln|w| = (1-n)\ln r + C \implies w(r) = \frac{c_1}{r^{n-1}}$$

So $v'(r) = \frac{c_1}{r^{n-1}}$. Integrating this, we have two cases based on dimension n:

- **For** $n = 2$: $v(r) = c_1 \ln r + c_2$.
- **For** $n \geq 3$: $v(r) = \frac{c_1}{(2-n)} \frac{1}{r^{n-2}} + c_2$.

2.3.3 Defining the Fundamental Solution

We choose the constants to normalize the total flux through a sphere to 1. The **fundamental solution** $\Phi(x)$ is defined as:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln|x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

2.3.4 Estimates of Derivatives

Let us calculate $D\Phi$ and $D^2\Phi$. For $n \geq 3$:

$$\Phi_{x_i} = \frac{1}{n(n-2)\alpha(n)}(2-n)|x|^{1-n}\frac{x_i}{|x|} = -\frac{1}{n\alpha(n)}\frac{x_i}{|x|^n}$$

Thus, $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$. For the second derivative:

$$\Phi_{x_i x_j} = -\frac{1}{n\alpha(n)}\left(\frac{\delta_{ij}}{|x|^n} - n\frac{x_i x_j}{|x|^{n+2}}\right)$$

Thus, $|D^2\Phi(x)| \leq \frac{C}{|x|^n}$.

2.4 Polar Coordinates, Volume, and Integrability

2.4.1 Polar Coordinates in Integration

For radial functions, it is highly useful to integrate in spherical coordinates. The volume of a ball $B(0,R)$ is $V_n(R) = \alpha(n)R^n$. The surface area of the sphere $\partial B(0,R)$ is denoted $S_n(R)$. Since the ball is the union of spheres of radius r from 0 to R:

$$V_n(R) = \int_0^R S_n(r)dr$$

Assuming $S_n(r) = cr^{n-1}$, we get $\alpha(n)R^n = \frac{c}{n}R^n$, so $c = n\alpha(n)$. Thus, the surface area is $n\alpha(n)R^{n-1}$. For any integrable function f, we can write:

$$\int_{B(0,R)} f(x)dx = \int_0^R \left(\int_{\partial B(0,r)} f dS\right) dr$$

2.4.2 Local Integrability Lemma

We need to understand when $1/|x|^\gamma$ is locally integrable around the singularity at $x = 0$.

$$\int_{B(0,1)} \frac{1}{|x|^\gamma} dx = \int_0^1 \left(\int_{\partial B(0,r)} \frac{1}{r^\gamma} dS \right) dr$$

$$= \int_0^1 \frac{1}{r^\gamma} \left(n\alpha(n) r^{n-1} \right) dr = n\alpha(n) \int_0^1 r^{n-1-\gamma} dr$$

This integral converges if and only if the exponent $n - 1 - \gamma > -1$, which means $\gamma < n$. **Conclusion:** - The fundamental solution $\Phi(x) \sim \frac{1}{|x|^{n-2}}$ is locally integrable because $n-2 < n$. - $D\Phi \sim \frac{1}{|x|^{n-1}}$ is locally integrable because $n - 1 < n$. - $D^2\Phi \sim \frac{1}{|x|^n}$ is **NOT** locally integrable because $n \not< n$.

2.5 Poisson's Equation and Convolution

To solve $-\Delta u = f$, we use the fundamental solution.

2.5.1 Definition of Convolution

The convolution of two functions f and g is defined as:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy$$

2.5.2 Commutativity of Convolution

We prove $(f * g)(x) = (g * f)(x)$. **In 1D:**

$$\int_{-\infty}^{\infty} f(x - y) g(y) dy$$

Let $z = x - y$. Then $y = x - z$ and $dy = -dz$. As $y \to \infty, z \to -\infty$ and $y \to -\infty, z \to \infty$.

$$= \int_{\infty}^{-\infty} f(z) g(x - z)(-dz) = \int_{-\infty}^{\infty} g(x - z) f(z) dz = (g * f)(x)$$

In $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} f(x - y) g(y) dy$$

Let $z = x - y$. The Jacobian determinant of this transformation is $\det(-I) = (-1)^n$. The change of variables formula for multiple integrals requires the absolute value of the Jacobian, which is $|(-1)^n| = 1$. Thus $dy = dz$.

$$= \int_{\mathbb{R}^n} f(z) g(x - z) dz = (g * f)(x)$$

2.5.3 The Difficulty

We propose $u = \Phi * f$. If we apply the Laplacian directly:

$$\Delta u(x) = \Delta \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy = \int_{\mathbb{R}^n} \Delta_x \Phi(x - y) f(y) dy$$

However, $\Delta\Phi = 0$ everywhere except at $x = y$ where it is undefined. We cannot simply pass the Laplacian inside because $|D^2\Phi(x)| \sim 1/|x|^n$, which is not integrable near 0. A more delicate proof is required.

2.6 Compact Support and Properties of Convolution

2.6.1 Compact Support

A function f has **compact support** if there exists a bounded set outside of which f is identically zero. The space of continuous functions with compact support is denoted $C_c(\mathbb{R}^n)$.

2.6.2 Support of Convolution

Theorem: If f is continuous and compactly supported, and g is locally integrable, then $(f * g)(x)$ is well-defined everywhere. If both have compact support, then $f*g$ has compact support. *Step-by-step Proof:* Fix x. The integrand $f(x-y)g(y)$ is non-zero only when $x-y \in \text{supp}(f)$ and $y \in \text{supp}(g)$. If f has compact support, say $f(z) = 0$ for $|z| > R$, then $f(x-y)$ is non-zero only for $|x-y| \leq R$. Thus, the integration over y is restricted to a bounded ball $B(x, R)$. Since g is locally integrable, the integral over this bounded set is finite, so $(f * g)(x)$ exists. If both have compact support, say $\text{supp}(f) \subseteq B(0, R_1)$ and $\text{supp}(g) \subseteq B(0, R_2)$, then for the integrand to be non-zero, we need $|x-y| \leq R_1$ and $|y| \leq R_2$. By triangle inequality:

$$|x| = |x-y+y| \leq |x-y| + |y| \leq R_1 + R_2$$

So if $|x| > R_1 + R_2$, the integral is strictly zero. Thus, the convolution has compact support contained in $B(0, R_1 + R_2)$.

2.6.3 Smoothness of Convolution

Theorem: If $f \in C^2(\mathbb{R}^n)$ and has compact support, and g is continuous, then $u = f * g \in C^2(\mathbb{R}^n)$. *Step-by-step Proof:* We write $u(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$. We want to show $\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} f_{x_i}(x-y)g(y)dy$. Consider the difference quotient:

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \frac{f(x+he_i-y) - f(x-y)}{h} g(y)dy$$

By the Mean Value Theorem, $\frac{f(x+he_i-y)-f(x-y)}{h} = f_{x_i}(x+\theta he_i - y)$ for some $\theta \in (0,1)$. Because f_{x_i} is continuous and has compact support, it is *uniformly continuous*. This means as $h \to 0$, $f_{x_i}(x+\theta he_i - y)$ converges to $f_{x_i}(x-y)$ **uniformly** in y. Because the convergence is uniform and the domain of integration is effectively bounded (since f has compact support), we can legitimately pass the limit inside the integral to obtain:

$$\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} f_{x_i}(x-y)g(y)dy$$

Applying the exact same argument to the first derivative, we get:

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} f_{x_i x_j}(x-y)g(y)dy$$

Alternative proof using Dominated Convergence Theorem (DCT): Let $F_n(y) = \frac{f(x+h_ne_i-y)-f(x-y)}{h_n} g(y)$ $F_n(y) \to f_{x_i}(x-y)g(y)$ point-wise as $h_n \to 0$. Since $f \in C^2$ and compactly supported, its derivative is globally bounded: $|f_{x_i}| \leq M$. Thus by MVT, $|F_n(y)| \leq M|g(y)|\chi_K(y)$ where K is a compact set slightly larger than the support of $f(x-\cdot)$. Since g is continuous on the compact set K, it is bounded, so $M|g(y)|\chi_K(y)$ is Lebesgue integrable. By the **Dominated Convergence Theorem**, we can interchange the limit and the integral:

$$\lim_{h_n \to 0} \int_{\mathbb{R}^n} F_n(y)dy = \int_{\mathbb{R}^n} \lim_{h_n \to 0} F_n(y)dy = \int_{\mathbb{R}^n} f_{x_i}(x-y)g(y)dy$$

Thus $u \in C^2$.