Real Analysis Exercise Sheet

Basic Topology

1. Let X be an infinite set. For $p, q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

   (a) Prove that this is a metric.

   (b) Which subsets of the resulting metric space are open?

   (c) Which are closed?

   (d) Which are compact? (Definition. A metric space (X, d) is called compact if every sequence in X has a convergent subsequence whose limit lies in X; a subset $Y \subset X$ is called compact if the subspace $(Y, d|_{Y\times Y})$ is compact).

   (e) Prove that a sequence $\{x_n\} \subset X$ converges to $x \in X$ in the discrete metric if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

2. Let (X, d) be a metric space. Show that $d'(x, y) = \sqrt{d(x, y)}$ is also a metric on X, and that the open sets for d' are the same as the open sets for d.

3. Consider $\mathbb{R}$ with the standard metric. Let $E \subset \mathbb{R}$ be a subset which has no limit points. Show that E is at most countable.

4. Let (X, d) be a compact metric space, and $f : X \to X$ a map such that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$. Prove that there exists a point x such that $f(x) = x$. Hint: how small can $d(x, f(x))$ get? Comment: this is an example of a fixed point theorem, very popular in various sciences for showing the existence of equilibria and such.

5. Let (X, d) be a metric space.

   (a) Prove the following:

      (i) For all $x, y, z \in X$
      $$|d(y, z) - d(z, x)| \leq d(x, y).$$

      (ii) For any $n \in \mathbb{N}$ and any $x_1, x_2, \ldots, x_n \in X$
      $$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

   (b) For $x \in \mathbb{R}$ and $y \in \mathbb{R}$ define

      (i) $d_1(x, y) = (x - y)^2$

      (ii) $d_2(x, y) = \sqrt{|x - y|}$

      (iii) $d_3(x, y) = |x^2 - y^2|$

      (iv) $d_4(x, y) = |x - 2y|$

   Determine, for each of these, whether it is a metric or not.

6. (a) In a metric space (X, d), prove that if a sequence (x_n) converges, then its limit is unique. That is, show that if $x_n \to x$ and $x_n \to y$, then $x = y$.

   (b) Prove that a subset E of a metric space (X, d) is closed if and only if
   $$E = \overline{E},$$
   where $\overline{E}$ denotes the closure of E.

7. Let $\mathbb{R}^2$ be equipped with the Taxicab metric/Manhattan metric defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

   (a) Prove that d is a metric.

(b) Draw the open balls $B_1((0,0))$, $B_2((0,0))$, and $B_{\frac{1}{2}}((0,0))$ in $\mathbb{R}^2$.

(c) On the same axes, sketch the corresponding Euclidean balls (under the usual metric) of the same radii.

(d) What geometric shape do the taxicab balls form?

(e) Are these taxicab balls open sets in the usual metric on Euclidean space? Prove that these two metrics are equivalent.

(f) Let

$$A = \{(x,y) \in \mathbb{R}^2 : d((x,y),(1,1)) \geq 2\}.$$

Sketch the set A. Is it closed in the taxicab metric?

8. (a) Let us consider the set $X = \mathbb{R} \setminus \{\sqrt{2}\}$ equipped with the usual metric $d(x,y) = |x-y|$. Is (X,d) a complete metric space? Justify your answer with appropriate reasoning.

(b) Let R be the set of real numbers with the usual metric $d(x,y) = |x-y|$.

(i) Show that the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence in R.

(ii) Show that the sequence $x_n = n$ is not a Cauchy sequence in R.

9. Let $C[0,1]$ be the set of all real-valued continuous functions on [0, 1]. Define:

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx.$$

Define a sequence $\{f_n\} \subset C[0,1]$ by:

$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ n(x - (\frac{1}{2} - \frac{1}{n})), & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

(a) Draw the graph of each function f_n and verify that $f_n \in C[0,1]$ by observing that each f_n is piecewise linear and continuous.

(b) Prove that f_n pointwise converge to f where

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

(c) Verify that $\{f_n\}$ is a Cauchy sequence in $(C[0,1], d_1)$.

(d) Is $(C[0,1], d_1)$ a complete metric space?

10. Let l^2 be the set of all sequences $x = \{x_n\}$ over the real numbers R such that $\sum_{n\geq 1} x_n^2$ converges. Prove that $d(x,y) = \sqrt{\sum_{n\geq 1}(x_n - y_n)^2}$ gives a metric on l^2.

11. Let (X,d) be a metric space and A be a non-empty subset of X. For $x \in X$, we define

$$d(x,A) = \inf\{d(x,a) : a \in A\}.$$

(a) Show that $d(x,A) = 0$ if and only if x lies in the closure of A.

(b) Show that if A is compact, then $d(x,A) = d(x,a)$ for some $a \in A$.

(c) The ϵ-neighborhood of A is defined to be

$$N(A;\epsilon) = \{x \in X : d(x,A) < \epsilon\}.$$

Prove that $N(A;\epsilon)$ is the union of the open balls $B_\epsilon(a)$ for $a \in A$.

(d) If A is compact and U is an open set containing A, prove that there exists $\epsilon > 0$ such that

$$A \subset N(A;\epsilon) \subset U.$$

Is this fact still true if A is just a closed set?

12. Let (X, d) be a metric space. Then verify the following properties:
    (a) The union of any collection $\{G_\alpha\}$ of open sets is open; that is, $\bigcup_\alpha G_\alpha$ is open.
    (b) The intersection of any collection $\{F_\alpha\}$ of closed sets is closed; that is, $\bigcap_\alpha F_\alpha$ is closed.
    (c) The intersection of any finite collection $G_1, \ldots, G_n$ of open sets is open; that is, $\bigcap_{i=1}^n G_i$ is open.
    (d) The union of any finite collection $F_1, \ldots, F_n$ of closed sets is closed; that is, $\bigcup_{i=1}^n F_i$ is closed.
13. In vector spaces, metrics are usually defined in terms of norms which measure the length of a vector. If V is a vector space defined over $\mathbb{R}$, then a norm is a function from vectors to real numbers, denoted by $||\cdot||$, satisfying:
    (i) $||x|| \geq 0$ and $||x|| = 0 \iff x = 0$,
    (ii) For any $\lambda \in \mathbb{R}$, $||\lambda x|| = |\lambda| ||x||$,
    (iii) $||x + y|| \leq ||x|| + ||y||$.

    Prove that every norm defines a metric.
14. Let M be a metric space with metric d. Show that d_1 defined by
    $$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$
    is also a metric on M. Observe that M itself is bounded in this metric.
15. Let A and B be two subsets of a metric space M. Recall that A°, the interior of A, is the set of interior points of A. Prove the following:
    $$\text{a)} A^\circ \cup B^\circ \subseteq (A \cup B)^\circ, \quad \text{b)} A^\circ \cap B^\circ = (A \cap B)^\circ$$
    Give an example of two subsets A and B of the real line such that $A^\circ \cup B^\circ \neq (A \cup B)^\circ$.
16. Let A be a subset of a metric space M. Recall that $\overline{A}$, the closure of A, is the union of A and its limit points. Recall that a point belongs to the boundary of A, ∂A, if every open ball centered at the point contains points of A and points of A^c, the complement of A. Prove that:
    (a) $\partial A = \overline{A} \cap \overline{A^c}$
    (b) $p \in \partial A \iff p$ is in $\overline{A}$, but not in A° (symbolically, $\partial A = \overline{A} \backslash A^\circ$)
    (c) ∂A is a closed set
    (d) A is closed $\iff \partial A \subseteq A$
17. Show that, in $\mathbb{R}^n$ with the usual (Euclidean) metric, the closure of the open ball $B_R(p)$, $R > 0$, is the closed ball
    $$\{q \in \mathbb{R}^n : d(p, q) \leq R\}.$$
    Give an example of a metric space for which the corresponding statement is false.
18. Prove directly from the definition that the set $K \subseteq \mathbb{R}$ given by
    $$K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\}$$
    is compact.
19. Let K be a compact subset of a metric space M, and let $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ be an open cover of K. Show that there is a positive real number δ with the property that for every $x \in K$ there is some $\alpha \in I$ with
    $$B_\delta(x) \subseteq \mathcal{U}_\alpha$$
20. Let M be a non-empty set, and let d be a real-valued function of ordered pairs of elements of M satisfying
    (a) $d(x, y) = 0 \iff x = y$

(b) $d(x, y) \leq d(x, z) + d(y, z)$.

Show that d is a metric on M.

21. Determine the boundaries of the following sets, $A \subseteq X$:

    (i) $A = \mathbb{Q}$, $X = \mathbb{R}$
    (ii) $A = \mathbb{R} \backslash \mathbb{Q}$ $X = \mathbb{R}$
    (iii) $A = (\mathbb{Q} \times \mathbb{Q}) \cap B_R(0)$ $X = \mathbb{R}^2$

22. Describe the interior of the Cantor set.

23. Let M be a metric space with metric d, and let d_1 be the metric defined above (in problem 2). Show that the two metric spaces (M, d), (M, d_1) have the same open sets.

24. Let A be a proper dense subset of $\mathbb{R}$, and let U be a non-empty open subset of $\mathbb{R}$. For each of the following statements, prove or disprove:

    (a) $U \subseteq \overline{A \cap U}$.
    (b) $\overline{A \cap U} = \varnothing$.
    (c) $\overline{A \cap U} \subseteq U$.
    (d) $\overline{A \cap U} = A \cap \overline{U}$.

25. Let $\{A_i : i \in I\}$ be a family of subsets of $\mathbb{R}$, where I is an index set. Prove or disprove the following statements:

    (a) If I is finite, then
    $$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$
    (b) If I is an arbitrary (possibly infinite) index set, then
    $$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$
    (c) For any index set I,
    $$\overline{\bigcap_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}.$$

26. Recall that a subset $S \subseteq \mathbb{R}^n$ is said to be *compact* if every open cover of S admits a finite subcover. With this in mind, answer the following:

    (a) Let $S \subseteq \mathbb{R}^n$ be unbounded. Construct an open cover of S that admits no finite subcover, and prove your claim.
    (b) Let $S \subseteq \mathbb{R}^n$ be not closed. Construct an open cover of S that admits no finite subcover, and prove your claim.

27. **(Discrete metric and compactness).** Let (X, d) be a metric space with the *discrete metric*
    $$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$
    Prove the following statements.

    (a) Every subset of X is both open and closed.
    (b) The space (X, d) is bounded (i.e., has finite diameter), regardless of the underlying set X.
    (c) (X, d) is compact if and only if X is finite. *Hint:* If X is infinite, consider the open cover by singleton sets.

(d) Conclude that in (X, d) the implication "closed and bounded $\Rightarrow$ compact" fails whenever X is infinite (since X is closed and bounded but not compact).

(e) Provide a concrete example with $X = \mathbb{N}$ under the discrete metric to illustrate the above points.

28. **Topology of the Cantor Set**

Let $C \subseteq [0, 1]$ denote the standard middle-third Cantor set.

Definitions:

- The *interior* of a set $A \subseteq \mathbb{R}$, denoted $\text{int}(A)$, is the largest open set contained in A.
- A space (or subset of a space) is called *totally disconnected* if its only connected subsets are singletons (sets consisting of a single point).
- A set A is called *self-similar* if it can be written as a union of scaled and translated copies of itself. For the Cantor set, this means

$$C = \tfrac{1}{3}C \cup \left(\tfrac{2}{3} + \tfrac{1}{3}C\right).$$

Questions:

(a) (**Closedness**) Show that C is a closed subset of $[0, 1]$.

(b) (**Interior**) Show that $\text{int}(C) = \varnothing$.

(c) (**Perfectness**) Show that every point of C is a limit point of C. (That is, C has no isolated points.)

(d) (**Total Disconnectedness**) Prove that the only connected subsets of C are singletons.

(e) (**Compactness**) Show that C is compact. (Hint: C is a closed subset of $[0, 1]$.)

(f) (**Product Topology** — do this only after product topology is covered in class) Show that C is homeomorphic to the product space $\{0, 1\}^{\mathbb{N}}$ with the product topology, where each copy of $\{0, 1\}$ is given the discrete topology.

(g) (**Self-similarity**) Show that C is self-similar, in the sense that

$$C = \tfrac{1}{3}C \cup \left(\tfrac{2}{3} + \tfrac{1}{3}C\right).$$

29. **Connected Subsets under Different Metrics**

Let (X, d) be a metric space. Recall that a subset $A \subseteq X$ is *connected* if it cannot be written as the union of two disjoint nonempty open subsets in the subspace topology.

Questions:

(a) (**Euclidean metric on** $\mathbb{R}$) Let $d(x, y) = |x - y|$.

i. Show that if $A \subseteq \mathbb{R}$ is connected, then for any $x, y \in A$ with $x < y$, the entire interval $[x, y]$ is contained in A.

ii. Deduce that the connected subsets of $(\mathbb{R}, d)$ are precisely the intervals (including singletons).

(b) (**Boundedness vs connectedness**) Give one example of a bounded connected subset of $\mathbb{R}$, and one example of an unbounded connected subset.

(c) (**Discrete metric on** $\mathbb{R}$) Consider the discrete metric

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Show that the only connected subsets of $(\mathbb{R}, d)$ are singletons.

(d) (**Truncated metric on** $\mathbb{R}$) Define $d(x, y) = \min\{1, |x - y|\}$ on $\mathbb{R}$.

i. Show that this is indeed a metric.

ii. Determine whether the connected subsets of $(\mathbb{R}, d)$ are the same as those under the Euclidean metric.

(e) (**Connectedness in $\mathbb{R}^n$**) Let $\mathbb{R}^n$ be equipped with the Euclidean metric.

i. Show that convex subsets of $\mathbb{R}^n$ are connected.

ii. Give an example of a connected subset of $\mathbb{R}^2$ that is not convex.

(f) (**Thought exercise**) Can you find (or prove impossible) a metric on $\mathbb{R}$ in which every infinite set is connected?

30. **Cardinality of Subsets of $\mathbb{R}^n$**

Let $n \geq 1$ and consider subsets of $\mathbb{R}^n$. For each statement, prove or provide a counterexample.

(a) (**Finite sets**)

i. Every finite subset of $\mathbb{R}^n$ is closed.

ii. Every finite subset of $\mathbb{R}^n$ is bounded.

(b) (**Countable sets**)

i. The set $\mathbb{Q}^n$ (vectors with rational coordinates) is countable.

ii. A countable subset of $\mathbb{R}^n$ can be dense.

(c) (**Uncountable sets**)

i. The interval $[0, 1]^n \subset \mathbb{R}^n$ is uncountable.

ii. Every uncountable subset of $\mathbb{R}^n$ contains a limit point. *(Hint: use Bolzano–Weierstrass theorem.)*

(d) (**Mixed cardinality statements — Prove or Disprove**)

i. The union of two countable sets is countable.

ii. The union of a countable set and an uncountable set is uncountable.

iii. Every subset of $\mathbb{R}^n$ is either countable or has the same cardinality as $\mathbb{R}$.

iv. Every infinite subset of $\mathbb{R}^n$ is uncountable.

v. The Cartesian product $\mathbb{Q} \times \mathbb{Q}$ is countable.

(e) (**Bonus**)

i. Show that $[0, 1] \subset \mathbb{R}$ and $[0, 1]^n \subset \mathbb{R}^n$ have the same cardinality.

ii. Show that $\mathbb{R}^n$ is uncountable for any $n \geq 1$.

Continuity

Before attempting the problems, keep the following points in mind:

1. The **limit** of a function is defined only at *limit points* of the domain. Therefore, the limit need not exist at all points of the domain and is not defined at isolated points. It should also be noted that even when the point p, at which we are finding the limit, belongs to the domain, the limit need not be equal to $f(p)$.

2. The limit of the quotient of two functions (Theorem 4.4) is valid only when the *limit of the denominator function* is nonzero. That is, $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is defined only if $\lim_{x \to a} g(x) \neq 0$.

3. **Continuity** of a function is defined at all points of its domain, including isolated points. Limit points outside the domain are not considered.

4. By definition of continuity, *any function is automatically continuous at an isolated point.* One can choose $\delta > 0$ small enough so that the neighborhood $N_\delta(a)$ contains only the point a from the domain.

5. At a limit point within the domain (Theorem 4.6), continuity requires that the limit coincides with the function value. Carefully distinguish between continuity at isolated points and at limit points.

6. **Continuity in metric and topological spaces:**

- **Metric space continuity (epsilon-delta definition):** For a function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y), f is continuous at $p \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d_X(x,p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon.$$
This definition is *distance-based*.
- **Topological space continuity (open set definition):** A function $f : X \to Y$ between topological spaces is continuous if the preimage of every open set in Y is open in X. This definition is *topology-based*, focusing on *openness* rather than distances.
- **Key comparisons:** Topological continuity is more general; metric spaces are a special case of topological spaces. In a metric space, the epsilon-delta and open set definitions are equivalent. The topological definition is more abstract, emphasizing the structure of open sets rather than specific distances.

7. Continuous functions map *compact sets* to compact sets. A continuous function from a compact set to $\mathbb{R}^k$ is bounded and attains its maximum and minimum. Moreover, every continuous function on a compact domain is *uniformly continuous.* To understand Theorem 4.20, consider examples when compactness is dropped; reconstruct the functions used in the proof with different examples to gain intuition.

8. Consider Example 4.21 carefully: Every point on the unit circle is of the form $(\cos t, \sin t)$. The map
$$t \mapsto (\cos t, \sin t)$$
takes $[0, 2\pi)$ to the unit circle centered at the origin.

   - If we start plotting from $t = 0$, the image begins at $(1, 0)$ and traverses the circle continuously until just before returning to $(1, 0)$ at $t = 2\pi$. The interval is kept open at 2π to avoid duplicating the starting point, making the map a bijection from $[0, 2\pi)$ onto the circle.
   - To analyze continuity, note that this map is continuous as a function from $[0, 2\pi)$ to $\mathbb{R}^2$, because each coordinate function $\cos t$ and $\sin t$ is continuous.
   - Now consider the inverse function, which maps points on the circle back to $[0, 2\pi)$:
   $$(\cos t, \sin t) \mapsto t.$$
   Take the point $(1, 0)$ on the circle, corresponding to $t = 0$. Consider an open disc of radius δ centered at $(1, 0)$ and intersect it with the circle.
     - Points in the first quadrant of the circle near $(1, 0)$ map to values of t close to 0.
     - Points in the fourth quadrant near $(1, 0)$ map to values of t close to 2π.
   - If we choose $\varepsilon < \pi$, there exist points in this δ-neighborhood whose images under the inverse are farther than ε apart. Hence, the inverse function is not continuous at $(1, 0)$. This example illustrates why compactness is important in Theorem 4.17: without compactness, a continuous bijection may fail to have a continuous inverse.

9. **Seminar Topic: Rearrangement of Series**

   - **Convergence of a series:** A series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $S_N = \sum_{n=1}^{N} a_n$ has a finite limit as $N \to \infty$.
   - **Absolute Convergence:** The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.
   - **Conditional Convergence:** The series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

   Theorem 3.54 (Rudin)

   Statement: If a series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then there exists a rearrangement of its terms such that the new series converges to any given real number M, or even diverges.

   Implication: This theorem highlights the delicate nature of conditionally convergent series; their sum can be altered by rearranging the terms.

   Riemann Series Theorem

- **Definition:** If a series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then for any real number M, there exists a rearrangement of the terms such that the new series converges to M, or even diverges.
- **Example:** Consider the alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln(2).$$

  By rearranging the terms appropriately, this series can be made to converge to any real number or diverge.
- **Significance:** This theorem underscores the importance of the order of terms in a conditionally convergent series.

Proof Sketch of Theorem 3.54

- **Idea:** Construct a rearrangement by selecting positive and negative terms such that the partial sums approach the desired limit.
- **Process:**
  (a) Begin with the original series $\sum_{n=1}^{\infty} a_n$.
  (b) Select enough positive terms so that the partial sum exceeds M.
  (c) Then, select enough negative terms so that the partial sum decreases below M.
  (d) Repeat this process, making the partial sums oscillate around M and converge to it.
- **Conclusion:** This construction demonstrates that a conditionally convergent series can be rearranged to converge to any real number M.

Importance of Absolute Convergence

- Absolute convergence ensures that the sum of a series is independent of the order of terms, whereas conditional convergence does not.
- **Theorem:** If a series is absolutely convergent, then any rearrangement of its terms converges to the same sum.
- **Contrast:** The Riemann Series Theorem does not apply to absolutely convergent series, highlighting the stability of absolutely convergent series under rearrangements.

31. *Limits and Limit Points:*

Consider the following functions and domains:

(a) Let $f : (0,1) \to \mathbb{R}$ be defined by $f(x) = x^2$. Show that $\lim_{x\to 0^+} f(x)$ exists even though $0 \notin \mathrm{Dom}(f)$.

(b) Let $g : [0,2] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2, & x \in [0,2), \\ 1, & x = 2. \end{cases}$$

Determine $\lim_{x\to 2^-} g(x)$ and verify that the limit exists but is not equal to $g(2)$.

(c) Let $h : (0,2) \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap (0,2), \\ 0, & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0,2). \end{cases}$$

Show that $\lim_{x\to 1} h(x)$ does not exist.

Instructions: For each part, identify the relevant limit point, determine if it belongs to the domain, and verify the existence or non-existence of the limit.

32. *Non-existence of Limits via Sequences:*
Let $f : (-1, 1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

(a) Consider the sequences $x_n = \frac{1}{n} > 0$ and $y_n = -\frac{1}{n} < 0$. Show that $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = -1$.

(b) Conclude that $\lim_{x\to 0} f(x)$ does not exist.

Hint: Use the sequential criterion for limits: if there exist sequences converging to the same point but with different limit values, then the limit at that point does not exist.

33. *Continuity on a Discrete Domain:*
Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by

$$f(n) = n^2, \quad n \in \mathbb{Z}.$$

(a) Identify the limit points of the domain $\mathbb{Z}$.

(b) Discuss the continuity of f at each point $n \in \mathbb{Z}$ using the ε–δ definition.

Hint: Recall that every point in a discrete set is isolated. By definition, any function is automatically continuous at isolated points. You may choose $\delta < 1$ to construct the ε–δ argument.

34. *Discontinuity via Sequences:*
(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is discontinuous at every point using the sequential criterion: take sequences of rational and irrational numbers converging to the same point.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x, & x \neq 1, \\ 2, & x = 1. \end{cases}$$

Use the sequential criterion to show that g is discontinuous at $x = 1$. Consider sequences $x_n \to 1$ with $x_n \neq 1$.

35. *Algebra of Limits and Continuous Functions:*
Let $f(x) = x^2$ and $g(x) = \sin x$, defined on $\mathbb{R}$.

(a) Using the algebra of limits, find

$$\lim_{x\to\pi} [f(x) + g(x)], \quad \lim_{x\to\pi} [f(x)g(x)], \quad \lim_{x\to\pi} \frac{f(x)}{1 + g(x)}.$$

(b) Prove directly (using the ε–δ definition) that the functions

$$h_1(x) = f(x) + g(x), \quad h_2(x) = f(x)g(x), \quad h_3(x) = \frac{f(x)}{1 + g(x)}$$

are continuous at $x = \pi$.

(c) State the general theorem about sums, products, and quotients of continuous functions and explain why it applies here.

Hint: For the quotient, ensure the denominator's limit at the point of interest is nonzero.

36. *Continuity of a Bijection and Its Inverse (True/False):*

Statement: There exists a continuous bijection $f : A \to B$, where $A, B \subset \mathbb{R}$, such that f is continuous but its inverse $f^{-1} : B \to A$ is not continuous.

Answer (True/False with justification):

Hint:Think about taking two disjoint intervals in the real line and mapping them continuously, piece by piece, onto a single connected interval. *On each piece the map can be something simple like a translation or the identity, so continuity is preserved locally. But when these two images meet at a common boundary point in the range, the inverse will "jump" between the two disjoint pieces of the domain - and that is exactly where the inverse loses continuity.*

For instance, this idea leads to

$$A = [0,1) \cup [2,3], \qquad B = [0,2], \qquad f(x) = \begin{cases} x, & x \in [0,1), \\ x-1, & x \in [2,3]. \end{cases}$$

37: *True/False Statements on Continuous Functions:*

Decide whether each of the following statements is **True** or **False**. Provide justification.

1. A continuous function maps open sets to open sets. *Hint: Think about $f(x) = x^2$ on $\mathbb{R}$. Is the image of an open interval always open?*

2. A continuous function maps bounded sets to bounded sets. *Hint: Consider $f(x) = x$ on $\mathbb{R}$. Is every bounded set mapped to a bounded set?*

3. A continuous function maps closed sets to closed sets. *Hint: Consider $f(x) = x$ on $(0,1)$. What happens to a closed set like $[0,1/2]$ if the domain is not closed in $\mathbb{R}$?*

4. A continuous real-valued function defined on a subset of $\mathbb{R}$ maps closed and bounded sets to closed and bounded sets. *Hint: Consider Heine–Borel theorem: closed and bounded sets in $\mathbb{R}$ are compact. Continuous functions map compact sets to compact sets.*

38. *Image of an Interval under a Continuous Map:*

Let $f : I \to \mathbb{R}$ be continuous, where $I \subset \mathbb{R}$ is an interval. Prove that $f(I)$ is also an interval.

Hint: Use the Intermediate Value Theorem (IVT): if f is continuous on I and $f(a) < y < f(b)$ for some $a, b \in I$, then there exists $c \in I$ such that $f(c) = y$. This shows that $f(I)$ contains all intermediate values between any two points, hence it is an interval.

39. *Monotone Functions and Discontinuities:*

(a) Monotone but Discontinuous: Give an example of a monotone function $f : [0,1] \to \mathbb{R}$ that is not continuous everywhere. *Hint: Consider a step function like* $f(x) = \begin{cases} 0, & 0 \le x < 1/2 \\ 1, & 1/2 \le x \le 1 \end{cases}$. *Verify monotonicity and identify the discontinuity.*

(b) Single Discontinuity: Let $f : [0,1] \to \mathbb{R}$ be a monotone function defined by

$$f(x) = \begin{cases} x, & x \neq 1/2, \\ 2, & x = 1/2. \end{cases}$$

Identify the point of discontinuity and verify that the set of discontinuities is countable.

(c) Countably Many Discontinuities: Construct a monotone function $g : [0,1] \to \mathbb{R}$ that has countably many discontinuities, for example, at all points of the form $x = 1/n$, $n \in \mathbb{N}$.

Definition (Characteristic Function): For a subset $A \subseteq X$, the characteristic function $\mathbf{1}_A : X \to \{0,1\}$ is defined as

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Hint: Consider

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}_{\{1/n \le x\}}(x),$$

where $\mathbf{1}_{\{1/n \leq x\}}$ is the characteristic function of the set $\{y \in [0,1] : y \geq 1/n\}$. Each step contributes a jump, the function is increasing, and it has countably many discontinuities.

40: *Limits at Infinity and Infinite Limits:*

(a) Limit is infinity at a finite point: Give an example of a function $f : (0,1) \to \mathbb{R}$ such that

$$\lim_{x \to 0^+} f(x) = +\infty.$$

Hint: Consider $f(x) = \frac{1}{x}$. *As* $x \to 0^+$, $f(x) \to +\infty$.

(b) Finite limit at infinity: Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to \infty} g(x) = L$$

for some finite $L \in \mathbb{R}$. *Hint: Consider* $g(x) = \frac{1}{x}$. *As* $x \to \infty$, $g(x) \to 0$.

41. **Lipschitz Continuity:** Let (X, d_X) and (Y, d_Y) be two **metric spaces**. A function $f : X \to Y$ is **Lipschitz continuous** if there exists a non-negative real number K (the **Lipschitz constant**) such that for all $x_1, x_2 \in X$:

$$d_Y(f(x_1), f(x_2)) \leq K \cdot d_X(x_1, x_2)$$

(For functions on $\mathbb{R}$, d_X and d_Y are replaced by the absolute difference, $|a - b|$).

Examples:

(a) $f(x) = 5x$ **on** $\mathbb{R}$:

$$|f(x_1) - f(x_2)| = 5|x_1 - x_2|$$

The constant is $\mathbf{K = 5}$.

(b) $f(x) = \sin(x)$ **on** $\mathbb{R}$: By the Mean Value Theorem, since $|\sin'(x)| = |\cos(x)| \leq 1$, we have:

$$|\sin(x_1) - \sin(x_2)| \leq 1 \cdot |x_1 - x_2|$$

The constant is $\mathbf{K = 1}$.

Question:

Does Lipschitz continuity imply uniform continuity?

Answer: Yes.

Hint: The definition allows you to choose δ as a constant multiple of ϵ ($\delta = \epsilon/K$), which works globally across the entire domain.

42. Is the **zero set** of a continuous function on a **metric space** always a **closed set**?

Zero Set Definition: Let (X, d_X) be a metric space, and let $f : X \to \mathbb{R}$ be a continuous function. The **zero set** of f is defined as: $Z(f) = \{x \in X \mid f(x) = 0\}$

(Topological Setup Consideration:) How does the topological characterization of continuity, specifically the property of the preimage of a closed set, immediately imply the answer?

43. **Sequential Criterion for Continuity:**Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function.

Definition (Sequential Criterion): f is **sequentially continuous** at $p \in X$ if for every sequence $\{x_n\} \subset X$ such that $x_n \to p$, the sequence of images $\{f(x_n)\}$ converges to $f(p)$, i.e.,

$$\lim_{n \to \infty} x_n = p \quad \Longrightarrow \quad \lim_{n \to \infty} f(x_n) = f(p).$$

The Question:

Is the standard $(\epsilon - \delta)$ definition of continuity equivalent to the Sequential Criterion (SC)?

That is, is f continuous at $p \iff f$ is sequentially continuous at p?

Hints for Equivalence:

(a) **Continuity $\implies$ SC:** Use the standard $(\epsilon - \delta)$ definition. The definition of a convergent sequence ensures that points are eventually close enough for the continuity condition to hold.

(b) **SC $\implies$ Continuity:** Use a proof by **contradiction**. If f is **not** continuous at p, the negation of the $(\epsilon - \delta)$ definition allows you to construct a sequence $\{x_n\}$ that converges to p but whose image sequence $\{f(x_n)\}$ fails to converge to $f(p)$.

44. **(Uniform Continuity and Extension)**Prove the following statement: A continuous function g on an open interval $(a, b) \subset \mathbb{R}$ is **uniformly continuous** on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function $\tilde{g}$ is continuous on the closed interval $[a, b]$.
Hint: To prove ($\implies$), show that uniform continuity implies the existence of the endpoint limits. For the limit $\lim_{x\to a^+} g(x)$, consider a sequence $\{x_n\} \subset (a, b)$ such that $x_n \to a$. Show that the sequence of images $\{g(x_n)\}$ is a **Cauchy sequence** in $\mathbb{R}$ by using the uniform continuity of g. Since $\mathbb{R}$ is complete, its limit exists. Use this limit to define the value $g(a)$. Finally, check the continuity of the extended function $\tilde{g}$ at a using the $\epsilon - \delta$ or sequential definition, potentially invoking the **triangle inequality** in the proof.