Real Analysis Notes

Course Code: MSMAT01DSC04

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1 Basic Topology

1.1 Finite, Countable and Uncountable Sets

Definition 1.1. *Let A and B be any two sets. Suppose there is a rule or correspondence that assigns to each element $x \in A$ exactly one element in B, which we denote by $f(x)$. We call this correspondence f a function from A to B, or a mapping of A into B.*

In this context, the set A is referred to as the domain of f (meaning f is defined on A), and the individual elements $f(x)$ are the values taken by f. The collection of all such values is called the range of f.

Definition 1.2. *Suppose we have a mapping f from a set A into a set B. For any subset $E \subseteq A$, we define $f(E)$ as the set of all elements of the form $f(x)$, where $x \in E$. We refer to $f(E)$ as the image of E under the function f.*

Notice that the entire range of f can be written as $f(A)$, and it naturally follows that $f(A) \subseteq B$. In the special case where $f(A) = B$ (meaning every element in the codomain B is mapped to by some element in A), we say that f maps A onto B. Such a function is also commonly called surjective.

Definition 1.3. *Let f be a mapping from a set A into a set B. For any subset $E \subseteq B$, we use the notation $f^{-1}(E)$ to denote the set of all elements $x \in A$ which map into E (that is, where $f(x) \in E$). This set $f^{-1}(E)$ is called the inverse image (or preimage) of E under f.*

Definition 1.4. *For a specific element $y \in B$, $f^{-1}(y)$ represents the set of all $x \in A$ such that $f(x) = y$. If, for every $y \in B$, the set $f^{-1}(y)$ contains at most one element from A, we say that f is a 1-1 (one-to-one) mapping from A into B.*

An equivalent and often more practical way to state this is: a function f is 1-1 (or injective) provided that distinct inputs yield distinct outputs. Formally, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for any $x_1, x_2 \in A$.

Definition 1.5. *When there exists a 1-1 mapping from a set A onto a set B (a bijection), we say that A and B can be placed in a 1-1 correspondence. We also express this by saying A and B share the same cardinal number, or simply that they are equivalent, denoted by $A \sim B$.*

This equivalence behaves as an equivalence relation because it naturally satisfies three key properties:

1. ***Reflexivity:*** *$A \sim A$ for any set A.*
2. ***Symmetry:*** *If $A \sim B$, then $B \sim A$.*
3. ***Transitivity:*** *If $A \sim B$ and $B \sim C$, then $A \sim C$.*

Definition 1.6. *Let J denote the set of all positive integers, and for any given positive integer n, let $J_n = \{1, 2, \ldots, n\}$ represent the set of the first n positive integers. For an arbitrary set A, we define its size as follows:*

(a) A is finite if there exists some n such that $A \sim J_n$. (By convention, the empty set is also considered finite).

(b) A is infinite if it is not a finite set.

(c) A is countable (sometimes referred to as enumerable or denumerable) if $A \sim J$.

(d) A is uncountable if it is neither finite nor countable.

(e) A is at most countable if it is either finite or countable.

Remark 1.7. *For two finite sets A and B, the equivalence $A \sim B$ clearly holds if and only if A and B contain the exact same number of elements. However, when dealing with infinite sets, simply "counting" elements to establish cardinality does not work, and the intuitive idea of "having the same number of elements" becomes quite vague. This is why the rigorous notion of a 1-1 correspondence is crucial; it allows us to mathematically establish when two infinite sets have the "same size" or "same number of elements" while retaining complete clarity.*

Example 1.8. *Let A be the set of all integers. We can show that A is countable by setting up a 1-1 correspondence with the set of positive integers J. Consider the following arrangement of the sets A and J:*

$$\begin{array}{rrrrrrrr} A: & 0, & 1, & -1, & 2, & -2, & 3, & -3, \dots \\ J: & 1, & 2, & 3, & 4, & 5, & 6, & 7, \dots \end{array}$$

In this case, we can provide an explicit formula for a function f from J to A which establishes this 1-1 correspondence:

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even,} \\ -\dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Example 1.9. *Any two open intervals (a, b) and (c, d) have the same cardinality. We can explicitly construct a bijection $f : (a, b) \to (c, d)$ using a linear transformation:*

$$f(x) = c + \frac{d-c}{b-a}(x - a)$$

This exact same mapping provides a 1-1 correspondence between any two closed intervals $[a, b]$ and $[c, d]$. (Note: For students' understanding, observe that these linear maps are continuous; they simply stretch and shift one interval to perfectly cover the other).

Example 1.10. *Any open interval has the same cardinality as the entire set of real numbers $\mathbb{R}$. For instance, consider the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$. A natural bijection is the tangent function:*

$$f(x) = \tan(x) \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(Note: This mapping is also continuous, extending the bounded interval to cover all of $\mathbb{R}$).

Example 1.11. *An open interval and a closed interval have the same cardinality, for instance $(0, 1) \sim [0, 1]$. An elegant way to prove this is by using the Schröder-Bernstein Theorem.*

Schröder-Bernstein Theorem (without proof): *If there exists a 1-1 mapping from a set A into a set B, and a 1-1 mapping from B into A, then there exists a bijection between A and B (i.e., $A \sim B$).*

To apply this to $(0, 1)$ and $[0, 1]$, we just need two 1-1 functions:

$$\begin{aligned} f &: (0,1) \to [0,1] \quad \text{defined by } f(x) = x \\ g &: [0,1] \to (0,1) \quad \text{defined by } g(x) = \frac{1}{4} + \frac{1}{2}x \end{aligned}$$

Since both f and g are 1-1, the theorem guarantees $(0, 1) \sim [0, 1]$.

Alternatively, we can construct an explicit bijection. Consider the sequence of points $1/2, 1/3, 1/4, \dots$ inside $(0, 1)$. We can define a function $h : (0, 1) \to [0, 1]$ that maps the first two terms of this sequence to the endpoints 0 and 1, shifts the rest of the sequence, and leaves everything else unchanged:

$$h(x) = \begin{cases} 0 & \text{if } x = 1/2 \\ 1 & \text{if } x = 1/3 \\ \frac{1}{n-2} & \text{if } x = 1/n \text{ for } n \geq 4 \ (n \in \mathbb{N}) \\ x & \text{if } x \neq 1/n \text{ for any integer } n \geq 2 \end{cases}$$

(Observe that unlike the previous examples, this piecewise explicit bijection is not continuous).

Remark 1.12. *In the first two examples, the bijections and their inverses are continuous. You will learn in topology that a continuous bijection with a continuous inverse is called a homeomorphism, which effectively identifies two spaces as having the "same shape".*

However, when mapping a closed interval to an open interval (as in our third example), you cannot get a homeomorphism. In fact, no bijection between an open and closed interval can be continuous in both directions. This is due to a topological property you will encounter later: continuous functions map compact sets (like closed, bounded intervals) to compact sets. Since an open interval is not compact, a continuous bijection from a closed interval to an open interval is impossible.

Remark 1.13. *Using the sequence-shifting technique or the Schröder-Bernstein theorem from the third example, 1-1 correspondences between open intervals and half-open intervals, or half-open intervals and closed intervals, can be established quite similarly.*

The ultimate takeaway from these examples is profound: the cardinality of the real numbers $\mathbb{R}$ *is exactly the same as the cardinality of any interval (open, closed, or half-open), provided the interval has a non-zero length.*

Remark 1.14. *A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is demonstrated by our previous example establishing a 1-1 correspondence between the positive integers* J *and all integers* A*, even though* J *is a proper subset of* A*.*

In fact, we could replace our earlier definition of an infinite set with the following statement: A set A *is infinite if* A *is equivalent to one of its proper subsets.*

Definition 1.15. *By a sequence, we mean a function* f *defined on the set* J *of all positive integers. If we evaluate this function at an integer* $n \in J$ *and let* $f(n) = x_n$*, it is customary to denote the sequence by the symbol* $\{x_n\}$*, or simply by listing its outputs:* $x_1, x_2, x_3, \ldots$ *.*

The values of f*, that is, the elements* x_n*, are called the terms of the sequence. If* A *is a set and if* $x_n \in A$ *for all* $n \in J$*, then* $\{x_n\}$ *is said to be a sequence in* A*, or a sequence of elements of* A*. Note that unlike sets, the terms* $x_1, x_2, x_3, \ldots$ *of a sequence need not be distinct.*

Remark 1.16. *The concept of a sequence gives us a highly intuitive way to understand countable sets. Since a countable set has a 1-1 correspondence with the positive integers* J*, every countable set can be viewed as the range of a sequence where all terms are distinct.*

In simple words, if a set is countable, its elements can be enumerated. This means they can be "arranged in a sequence." You can literally line the elements up and label them: "Here is the first element (x_1*), here is the second element (*x_2*), here is the third (*x_3*)," and so on. You are guaranteed that every single element in the set will eventually appear on your list exactly once.*

(As a minor note, sometimes it is more convenient to replace J *in our definition with the set of all non-negative integers, meaning we simply start our sequence index with* 0 *rather than* 1*).*

Theorem 1.17. *Every infinite subset of a countable set* A *is countable.*

Proof. Suppose $E \subset A$, and E is an infinite set. Our goal is to show that we can arrange the elements of E into a sequence, proving it is countable.

Step 1: List the elements of the parent set. Since A is countable, we know we can arrange all of its elements into a single sequence of distinct terms: $x_1, x_2, x_3, \ldots$.

Step 2: Pick out the elements of E in order. We will build a sequence just for E by scanning through our list of A and picking out the elements that belong to E. We do this by keeping track of their index positions:

- Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. This is the first element of E we encounter.
- Having found the first element, let n_2 be the next smallest integer (greater than n_1) such that $x_{n_2} \in E$.
- In general, once we have found $k-1$ elements (at positions $n_1, \ldots, n_{k-1}$), we find the next element by letting n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Step 3: Establish the 1-1 correspondence. Because E is an infinite set, this scanning process will never stop; we will never run out of elements. We can define a function f from the set of positive integers J to E by $f(k) = x_{n_k}$ for $k = 1, 2, 3, \ldots$.

This function matches every positive integer k to exactly one distinct element of E, exhausting all elements of E in the process. This establishes a perfect 1-1 correspondence between J and E, meaning E is countable. $\square$

Remark 1.18. *This theorem reveals a fundamental property of infinite sets: roughly speaking, countable sets represent the "smallest" type of infinity. Because any infinite subset of a countable set simply drops down to being countable again, it is impossible for an uncountable set to exist as a subset of a countable set.*

Definition 1.19. *Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α. The set whose elements are these subsets E_α is denoted by $\{E_\alpha\}$. Instead of confusingly saying "a set of sets," we generally refer to this as a collection of sets or a family of sets.*

For any such family of sets, we can generalize the operations of union and intersection:

- ***Arbitrary Union:*** *The union of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation:*

$$S = \bigcup_{\alpha \in A} E_\alpha$$

- ***Arbitrary Intersection:*** *The intersection of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation:*

$$P = \bigcap_{\alpha \in A} E_\alpha$$

Finally, if $A \cap B$ is not empty, we say that A and B intersect; if it is empty, they are disjoint.

Remark 1.20. *To make sense of this notation, it helps to think of the set A as an "index set" (like a collection of name tags). The way we write unions and intersections changes depending on how large this index set is:*

- ***Finite Collections:*** *If A just consists of the integers $1, 2, \ldots, n$, we are dealing with a finite number of sets. We usually write the union explicitly as $S = E_1 \cup E_2 \cup \cdots \cup E_n$, or compactly using a limit: $S = \bigcup_{m=1}^{n} E_m$. The exact same logic and notation apply to finite intersections, replacing the U with $\cap$.*

- ***Countable Infinite Collections:*** *If A is the set of all positive integers, we have an infinite sequence of sets. The usual notation for their union becomes $S = \bigcup_{m=1}^{\infty} E_m$. It is important to note that the ∞ symbol here simply indicates that we are taking the union of a countable collection of sets; it should not be confused with the literal numbers $+\infty$ or $-\infty$.*

- ***Uncountable Arbitrary Collections:*** *The generalized notation $\bigcup_{\alpha \in A} E_\alpha$ is incredibly powerful because it works even if A is an uncountable set (like the real numbers). It guarantees that no matter how unimaginably large the family of sets is, we can rigorously define what it means to combine them all together or find what they all share.*

Example 1.21. *Let A be the set of real numbers x such that $0 < x \leq 1$. We can write this index set as $A = (0, 1]$.*

For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. In other words, each set E_x is the open interval $(0, x)$.

Then this family of sets has the following properties:

(i) $E_x \subset E_z$ *if and only if* $0 < x \leq z \leq 1$;

(ii) $\displaystyle\bigcup_{x \in A} E_x = E_1$;

(iii) $\displaystyle\bigcap_{x \in A} E_x$ *is empty.*

Explanation of Properties:

- ***Property (i)*** *is visually clear. The open interval $(0, x)$ is completely contained inside the interval $(0, z)$ if and only if the upper bound x is less than or equal to z. This shows our sets are "nested" inside one another.*

- ***Property (ii)*** *is also straightforward. Since the sets are nested, as x gets larger, the interval E_x gets larger. The union of all these intervals is simply the largest interval in the collection. Because x can take a maximum value of 1 (since $x \in A$), the largest set is $E_1 = (0, 1)$. Every element in the union will fall into this maximal set.*

- ***To prove Property (iii)****, we must show that no single real number belongs to every set E_x. Suppose, for the sake of contradiction, that there is some real number y in the intersection. For y to be in any E_x, it must be strictly positive ($y > 0$).*

  However, no matter how tiny this positive number y is, we can always find a real number $x \in A$ that is strictly smaller than y (for instance, let $x = y/2$). For this specific choice of x, y is not in E_x because y is not less than x. Since $y \notin E_x$ for this particular x, it cannot be in the intersection of all E_x. Thus, the intersection must be empty.

Theorem 1.22. *Let $\{E_n\}$, $n = 1, 2, 3, \dots$, be a sequence of countable sets, and put*

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Before diving into the formal proof, it is helpful to state what this theorem means in simple terms: **a countable union of countable sets is countable.**

Proof. Since each individual set E_n is countable, its elements can be arranged in a sequence. Let us denote the elements of the first set E_1 as $x_{11}, x_{12}, x_{13}, \dots$, the elements of the second set E_2 as $x_{21}, x_{22}, x_{23}, \dots$, and in general, the elements of the n-th set E_n as $\{x_{nk}\}$ for $k = 1, 2, 3, \dots$.

Step 1: Construct an infinite grid.
We can arrange all the elements of all the sets into an infinite two-dimensional array, where the n-th row contains all the elements of the set E_n:

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

This grid contains every single element of the union set S.

Step 2: Traverse the grid to form a single sequence.
To show that S is countable, we must show that all elements in this grid can be listed in a single, one-dimensional sequence without missing any. We cannot just read across the first row, because it is infinitely long and we would never reach the second row.

Instead, we read the grid by tracing finite diagonals (moving from bottom-left to top-right along diagonals). This gives us the sequence:

$$x_{11}; \quad x_{21}, x_{12}; \quad x_{31}, x_{22}, x_{13}; \quad x_{41}, x_{32}, x_{23}, x_{14}; \quad \dots$$

Step 3: Address duplicate elements.
The sets E_n might not be mutually disjoint; they might share elements. If they do, some elements will appear more than once in our new diagonal sequence.

If we move through our sequence and simply skip any element we have already listed, we create a 1-1 correspondence between the unique elements of S and a subset (let's call it T) of the positive integers. This establishes that S is *at most countable*.

Step 4: Conclude using previous results.
We know that the first set E_1 is completely contained within S (that is, $E_1 \subset S$). Since E_1 is an infinite set, S must also be an infinite set.

From Step 3, S is equivalent to T, which is a subset of the positive integers. Since S is infinite, T must be an *infinite* subset of the positive integers. By our previous theorem, **every infinite subset of a countable set is countable**. Because T is an infinite subset of the countable set of positive integers, T is countable, and therefore S is countable. □

Corollary 1.22.1. *Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put*

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable.

Proof. The text provides a very compact proof for this, stating that T is equivalent to a subset of the previously defined set S from equation 15. Here is how that deduction works.

The previous theorem proved that a strictly countable union of strictly countable sets is countable. This corollary relaxes those conditions to "at most countable," meaning the index set A might be finite, and the individual sets B_α might also be finite or even empty.

Because A is at most countable, we can arrange its elements into a sequence $\alpha_1, \alpha_2, \ldots$. This allows us to list our sets as $B_{\alpha_1}, B_{\alpha_2}, \ldots$. If A is finite, this list eventually stops. We can easily turn it into an infinite sequence by simply defining all the remaining sets in the sequence as empty sets. Adding empty sets to a union does not change the final set T.

Similarly, because each B_α is at most countable, it is either equivalent to a fully countable set or it is a subset of one. Therefore, when we take the union of all these B_α sets to form T, we are essentially building a set that is contained entirely within the union S from equation 15.

Because the previous theorem definitively proved that the larger set S is countable, and T is merely a subset of S, it logically follows that T must be at most countable. □

Theorem 1.23. *Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where $a_k \in A$ $(k = 1, \ldots, n)$, and the elements $a_1, \ldots, a_n$ need not be distinct. Then B_n is countable.*

Proof. We can prove this theorem using the principle of mathematical induction.

First, consider the base case where $n = 1$. The set B_1 consists of all 1-tuples, which are simply the individual elements of A. Since B_1 is exactly the same set as A, and we are given that A is countable, it is evident that B_1 must also be countable.

Next, we move to the inductive step. We assume that the theorem holds true for some integer $n-1$. This means we assume the set B_{n-1}, which contains all $(n-1)$-tuples, is a countable set. We must show that this assumption implies the set B_n is also countable.

Any n-tuple in B_n can be constructed by taking an $(n-1)$-tuple and appending one more element to it. Therefore, an element in B_n can be written in the form (b, a), where b represents an entire $(n-1)$-tuple from the set B_{n-1}, and a is a single element from our original set A.

Let us fix one specific $(n-1)$-tuple, which we will call b. We can form a collection of pairs by keeping this b constant and pairing it with every possible element a from the set A. Because the set A is countable, this specific collection of pairs (b, a) is equivalent in size to A, making it a countable set itself.

To construct the entire set B_n, we must perform this pairing process for every single $(n-1)$-tuple b that exists in B_{n-1}. We are essentially taking a union of all these individual, countable collections of pairs. The number of these collections is determined by the number of elements in B_{n-1}. Since our induction hypothesis established that B_{n-1} is a countable set, we are combining a countable number of these collections.

Consequently, B_n is formed by taking a countable union of countable sets. As established by our previous theorem, a countable union of countable sets is always countable. Thus, B_n is countable, and the proof follows by induction. □

Corollary 1.23.1. *The set of all rational numbers is countable.*

Proof. The text provides a concise proof by referencing our previous theorem about n-tuples. Here is how that connection works in detail.

Recall that a rational number is defined as any number that can be expressed as a fraction b/a, where both b and a are integers (with a not equal to zero). This means every single rational number can be represented by a pair of integers (b, a).

Our previous theorem established that if a base set is countable, then the set of all n-tuples formed from that base set is also countable. If we let our base set be the set of all integers, which we already proved is a countable set, and we set $n = 2$, the theorem guarantees that the set of all possible integer pairs (b, a) is countable.

Because every rational number is formed by one of these integer pairs, the set of all fractions b/a is essentially a subset of this larger set of pairs. We simply discard the pairs where the denominator

a is zero, and we recognize that some pairs represent the same fraction (like 1/2 and 2/4). Since the entire collection of integer pairs is countable, this sub-collection of valid fractions must also be countable. Therefore, the set of all rational numbers is countable. □

Exercise 1. *Prove that the set of all algebraic numbers is countable.*

Proof. Before proving the statement, we must first define what an algebraic number is. An algebraic number is any complex number x that is a root of a non-zero polynomial equation with integer coefficients. In other words, x is a solution to an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where the degree $n \geq 1$, all the coefficients $a_0, a_1, \ldots, a_n$ are integers, and the leading coefficient a_n is not zero.

To prove that the set of all such numbers is countable, we can organize them by the degree n of their corresponding polynomials. Let us fix a specific degree n. A polynomial of this degree is completely determined by its coefficients, which we can list as an $(n+1)$-tuple of integers: $(a_0, a_1, \ldots, a_n)$.

Because the set of all integers is countable, our earlier theorem regarding tuples guarantees that the set of all possible $(n+1)$-tuples of integers is also countable. Therefore, there are only countably many polynomials of degree n with integer coefficients.

Furthermore, a fundamental theorem of algebra states that any polynomial of degree n has at most n distinct roots. This means each polynomial only yields a finite set of algebraic numbers. To gather all algebraic numbers associated with degree n, we take the union of the roots from each of these polynomials. Since we have a countable number of polynomials, and each has a finite number of roots, we are taking a countable union of finite sets. This guarantees that the set of all algebraic numbers of a fixed degree n is countable.

Finally, to construct the entire set of all algebraic numbers, we must combine the sets of roots for every possible degree $n = 1, 2, 3, \ldots$. Since the set of algebraic numbers for each individual degree n is countable, combining all of them together means we are taking a countable union of countable sets. As established by our previous theorem, a countable union of countable sets is always countable. Thus, the set of all algebraic numbers is countable. □

Theorem 1.24. *Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable. (Note: The elements of A are sequences like $1, 0, 0, 1, 0, 1, 1, 1, \ldots$).*

Proof. We will prove this by showing that no countable list can possibly contain every sequence in A.

Step 1: Assume a countable subset.

Let E be any countable subset of A. Because E is countable, we can enumerate its elements in a sequence: $s_1, s_2, s_3, \ldots$. Keep in mind that each element s_n in this list is itself an infinite sequence of 0s and 1s.

Step 2: Construct a new sequence (Cantor's Diagonal Process).

We will now construct a brand new sequence, which we will call s. We determine the digits of s one by one using a famous technique known as **Cantor's diagonal process**:

- Look at the 1st digit of the first sequence s_1. If it is a 1, make the 1st digit of s a 0. If it is a 0, make it a 1.
- Look at the 2nd digit of the second sequence s_2. If it is a 1, make the 2nd digit of s a 0, and vice versa.
- In general, for any integer n, look at the n-th digit of the sequence s_n. If that digit is a 1, we define the n-th digit of our new sequence s to be 0, and vice versa.

Step 3: Show the new sequence is missing from the list.

Because our new sequence s consists entirely of 0s and 1s, it is clearly a valid element of the overall set A. But is it in our countable subset E? By our construction, s differs from the first sequence s_1 in the first digit. It differs from the second sequence s_2 in the second digit. In general, it differs from every sequence s_n in our list at the n-th digit. Therefore, s is not anywhere in the set E.

Step 4: Conclusion.

Since $s \in A$ but $s \notin E$, it means E is missing at least one element of A. Thus, E is a *proper subset* of A.

We have just shown that *every* countable subset of A is merely a proper subset of A. It logically follows that A itself must be uncountable. (If A were countable, we could let our list E be the entirety of A. But that would mean A is a proper subset of itself, which is absurd). □

Exercise 2. *Prove that the set of all real numbers $\mathbb{R}$ is uncountable.*

Proof. We will prove this by contradiction, relying on a principle established earlier: every infinite subset of a countable set must itself be countable.

Assume, for the sake of contradiction, that the entire set of real numbers $\mathbb{R}$ is countable.

Let us look at a very specific subset of the real numbers. Let A be the set of all real numbers between 0 and 1 whose decimal representations consist exclusively of the digits 0 and 1 (for example, numbers like $0.1011001\ldots$ or $0.0101010\ldots$). Clearly, A is a subset of $\mathbb{R}$.

Notice that there is a perfect 1-1 correspondence between the numbers in set A and the set of all infinite sequences made up of the digits 0 and 1. We proved in our previous theorem (using Cantor's diagonal process) that the set of all such 0 and 1 sequences is uncountable. Because they map perfectly to one another, our subset A must also be an uncountable set.

This creates a fatal logical flaw. Under our assumption that $\mathbb{R}$ is countable, every infinite subset of $\mathbb{R}$ must also be countable. However, we just constructed a subset A that is strictly uncountable. This contradiction proves that our initial assumption was false.

Therefore, the set of all real numbers $\mathbb{R}$ is uncountable. □

Remark 1.25. *This profound result leads to several important consequences regarding the "size" of different collections of numbers on the real line.*

Every interval is uncountable: *It is a known mathematical fact that any continuous interval of real numbers (such as the open interval $(0,1)$ or any interval (a,b)) has the exact same cardinality as the entire set of real numbers $\mathbb{R}$. Since we just proved that $\mathbb{R}$ is uncountable, it immediately shows that every interval on the number line is also uncountable.*

The irrational numbers are uncountable: *The set of real numbers $\mathbb{R}$ is simply the union of the rational numbers (which we denote as $\mathbb{Q}$) and the irrational numbers. We have already proven that the rational numbers $\mathbb{Q}$ are countable.*

If the set of irrational numbers were also countable, then $\mathbb{R}$ would be formed by taking the union of two countable sets. We know that a union of countable sets is always countable, which would mean $\mathbb{R}$ is countable. But $\mathbb{R}$ is uncountable! The only way to avoid this contradiction is if the set of irrational numbers is uncountable.

Intervals are dominated by irrationals: *We can apply this exact same logic to any specific segment of the number line. Take any interval I. The rational numbers that happen to fall inside this interval, denoted as $I \cap \mathbb{Q}$, simply form a subset of all rational numbers, meaning they are countable.*

The rest of the numbers in the interval are irrational. If the irrationals in the interval were countable, the entire interval I would be the union of two countable sets, making the interval itself countable. Since we know every interval is uncountable, its irrational part must be uncountable. In simple terms, within any interval, the rational numbers are merely a countable dusting, while the irrational numbers make up the uncountable bulk.