Real Analysis Assignment Sheet

The exam based on this is on 29 December 2025

1. Given two real numbers $a < b$, let $d(x, [a, b]) = \min\{|x - y| : a \leq y \leq b\}$ for any $-\infty < x < \infty$. Then the function

$$f(x) = \frac{d(x, [0, 1])}{d(x, [0, 1]) + d(x, [2, 3])}$$

satisfies

   (a) $0 \leq f(x) < \frac{1}{2}$ for every x

   (b) $0 < f(x) < 1$ for every x

   (c) $f(x) = 0$ if $2 \leq x \leq 3$ and $f(x) = 1$ if $0 \leq x \leq 1$

   (d) $f(x) = 0$ if $0 \leq x \leq 1$ and $f(x) = 1$ if $2 \leq x \leq 3$

2. Prove that a sequence $\{x_n\} \subset X$ converges to $x \in X$ in the discrete metric if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

3. Let $\mathbb{R}^2$ be equipped with the Taxicab metric/Manhattan metric defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

   (a) Prove that d is a metric.

   (b) Are these taxicab balls open sets in the usual metric on Euclidean space? Prove that these two metrics are equivalent.

4. Let ℓ^2 be the set of all sequences $x = \{x_n\}$ over the real numbers $\mathbb{R}$ such that

$$\sum_{n \geq 1} x_n^2 \text{ converges.}$$

Prove that

$$d(x, y) = \sqrt{\sum_{n \geq 1} (x_n - y_n)^2}$$

gives a metric on ℓ^2.

5. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$, and equip A with the subspace metric induced from the usual metric on $\mathbb{R}$. Let (Y, d_Y) be any metric space. Show that any function $f : A \to Y$ is continuous.

6. Give an example of a metric space (X, d) such that the following hold:

   (a) There exists a subset $A \subsetneq X$ with $\text{int}(\partial A) = X$.

   (b) There exist subsets $A, B \subseteq X$ such that $\text{int}(A) = \text{int}(B) = \emptyset$ but $\text{int}(A \cup B) = X$.

7. Let (X, d) be a compact metric space, and let $(x_{m,n})_{m,n\in\mathbb{N}}$ be a double sequence in X. Show that there exist increasing sequences of indices $m_1 < m_2 < \cdots$, $n_1 < n_2 < \cdots$ such that the diagonal subsequence

$$y_k := x_{m_k,n_k}, \quad k \in \mathbb{N},$$

converges in X.

8. Let $B = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$, equipped with the subspace metric from the usual metric on $\mathbb{R}$.

   (a) Define a function $g : B \to \mathbb{R}$ by $g(\frac{1}{n}) = n$ for all $n \in \mathbb{N}$ and $g(0) = 2$. Show that g is not continuous at 0.

   (b) Prove that a function $f : B \to \mathbb{R}$ is continuous at 0 if and only if $\lim_{x\to 0, x\in B} f(x) = f(0)$.

   (c) Let $f : \mathbb{R} \to \mathbb{Z}$ be a continuous function. Show that f must be a constant function.

9. Let (X, d) be a metric space. Prove or disprove the following:

   (a) For any subset $A \subseteq X$, we have $X \setminus \overline{(X \setminus A)} = \text{int}(A)$.

   (b) For any subset $A \subseteq X$, we have $\text{int}(A) = \text{int}(\overline{\text{int}(A)})$.

   (c) For any subset $A \subseteq X$, we have $\overline{\text{int}(A)} = \text{int}(\overline{\text{int}(A)})$.

10. Prove or disprove the followings:

    (a) In a metric space (X, d), the closure of an open ball $B_r(a) = \{x \in X : d(x, a) < r\}$ is the closed ball $\{x \in X : d(x, a) \leq r\}$.

    (b) Countable union of closed set is closed.

11. Let (x_n) be a sequence converging to a real number x. Assume that $a \leq x_n \leq b$ for some real numbers a and b. Show that $a \leq x \leq b$.

12. Let $0 < x_1 < 1$ and

$$x_n := \frac{1}{7}\left(x_{n-1}^3 + 2\right) \quad \text{for all } n \geq 2.$$

Show that the sequence x_n converges and the limit is a root of the equation

$$x^3 - 7x + 2 = 0.$$

13. Let x_n be a Cauchy sequence such that $x_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Show that there exists $k \in \mathbb{N}$ such that x_n is constant for all $n \geq k$.

14. Let $x_n \to x$ and $y_n \to y$. Then show that

$$\frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} \to xy.$$

15. Let a_n be a sequence such that $\sum_n a_n$ converges to a real number s. Show that the series $\sum_n (a_{2n} + a_{2n-1})$ converges and find its sum.

16. Let a_n be a sequence such that $a_n \to 0$. Show there exists a subsequence a_{n_k} such that the series $\sum_k a_{n_k}$ is convergent.

17. If $\lim_{n\to\infty} x_n = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

18. Find the limits of the following sequences:

    (a) $\lim\left((2+1/n)^2\right)$

    (b) $\lim\left(\frac{(-1)^n}{n+2}\right)$

    (c) $\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right)$

    (d) $\lim\left(\frac{n+1}{n\sqrt{n}}\right)$

19. Prove the Urysohn Subsequence Principle: Let $\{x_n\}$ be a sequence of real numbers with the following property: every subsequence of $\{x_n\}$ has a further subsequence which converges to x. Then show that sequence $\{x_n\}$ converges to x.

20. Show that if $c > 1$, then the following series are convergent:

    (a) $\sum \frac{1}{n(\ln n)^c}$

    (b) $\sum \frac{1}{n(\ln n)(\ln \ln n)^c}$

21. Let $0 < \alpha < 1$ and $f : [a, b] \to \mathbb{R}$ be a continuous function such that for each $x \in [a, b]$ there exists $y \in [a, b]$ such that $|f(y)| \leq \alpha |f(x)|$. Show that there exists a point $c \in [a, b]$ such that $f(c) = 0$.

22. Let (X, d) be a metric space and let $A \subset X$ be a finite nonempty set with the property that every $a \in A$ is a limit point of $X \setminus \{a\}$ (equivalently, no point of A is isolated in X).

    Construct functions $f, g : X \to \mathbb{R}$ such that

    (a) f and g are discontinuous at every point of A and continuous at every point of $X \setminus A$; and

    (b) both $f + g$ and fg are continuous on all of X.

    Give the construction and prove the claims.

23. **True/False Explain.** The function $f : \mathbb{R}^n \to \mathbb{R}$, defined as $f(x_1, \ldots, x_n) = \max\{|x_i|\}, i = 1, \ldots, n$, is uniformly continuous.

24. Verify the followings

    (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that its derivative f' is bounded; i.e., there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Using the Mean Value Theorem, prove that f is uniformly continuous on $\mathbb{R}$.

    (b) Consider the function $f(x) = \tan^{-1}(x)$. Verify that it satisfies the condition of Part (a) and confirm that it is uniformly continuous on $\mathbb{R}$.

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function such that $f(1/n) = 0$ for every positive integer n. Choose the correct statement(s) from below:

    (a) $f(0) = 0$;

    (b) $f'(0) = 0$;

    (c) $f''(0) = 0$;

    (d) f is a nonzero polynomial.

26. (a) Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) \to \ell$ as $x \to \infty$. Show that

    $$\frac{f(x)}{x} \to \ell \quad \text{as } x \to \infty.$$

(b) Let $f : [0, 1] \to [0, 1]$ be such that

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y| \quad \text{for all } x, y \in [0, 1].$$

Let

$$A = \{x \in [0, 1] \mid f(x) = x\}.$$

What is the number of elements in A?

27. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions, and suppose that $g(x) > 0$ for all $x \in [a, b]$. Prove that there exists a point $c \in [a, b]$ such that

$$\frac{1}{\int_a^b g(x)\,dx} \int_a^b f(x)g(x)\,dx = f(c).$$

28. Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \int_{-\infty}^{x} f(t)\,dt$. Choose the correct statement(s):

(a) f is continuous;

(b) F is continuous;

(c) F is uniformly continuous;

(d) There exists a positive real number M such that $|f(x)| < M$ for all $x \in \mathbb{R}$.

29. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that f and its derivative have no common zero in the closed interval $[0, 1]$. Show that f cannot have infinitely many zeroes in $[0, 1]$.

30. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that for any $r_1, r_2, r_3 \in f([0, 1])$ there exists $x \in [0, 1]$ such that

$$f(x) = \frac{r_1 + r_2 + r_3}{3}.$$

31. Let A, B be two non-empty disjoint closed subsets of a metric space (X, d). Prove that there exists a continuous function $f : X \to \mathbb{R}$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

32. Let $f \in C([0, 1])$ satisfy the symmetry $f(x) = f(1 - x)$ for all $x \in [0, 1]$. Show there exists a sequence of symmetric polynomials p_n (i.e. $p_n(x) = p_n(1 - x)$ for all x) with $p_n \to f$ uniformly on $[0, 1]$.

33. Suppose f is Riemann integrable on $[0, a]$ and define $g : [0, \lambda a] \to \mathbb{R}$ by $g(x) = f(x/\lambda)$ where $\lambda > 0$. Show that g is Riemann integrable on $[0, \lambda a]$ and

$$\int_0^{\lambda a} g\,dx = \lambda \int_0^a f\,dx.$$

34. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that
$$\int_a^b f(x)g(x)\,dx = 0$$
for all continuous functions $g : [a, b] \to \mathbb{R}$. Show that $f(x) = 0$ for all $x \in [a, b]$.

35. Let $g : [0, \frac{1}{2}] \to \mathbb{R}$ be a continuous function. Define a sequence of functions $\{g_n\}$ by
$$g_1 = g, \quad \text{and} \quad g_{n+1}(t) = \int_0^t g_n(s)\,ds, \quad \text{for all } n \geq 1.$$
Show that
$$\lim_{n\to\infty} n!\, g_n(t) = 0,$$
for all $t \in [0, \frac{1}{2}]$.

36. Let $f_n(x) = \frac{nx}{1+n^2x^p}$ on $[0, 1]$, with parameter $p > 0$. For which values of p does f_n converge uniformly to its pointwise limit?

37. Let the sequence of functions $f_n : [0, 1] \to \mathbb{R}$, $n \geq 2$, be given by
$$f_n(x) = \begin{cases} n^2x, & 0 \leq x \leq 1/n, \\ 2n - n^2x, & 1/n < x < 2/n, \\ 0, & 2/n \leq x \leq 1. \end{cases}$$
Choose the correct statement(s) from below:
   (a) f_n converges pointwise but not uniformly as $n \to \infty$.
   (b) f_n converges uniformly as $n \to \infty$.
   (c) The functions $\{f_n\}_{n\geq1}$ are not equicontinuous.
   (d) $\int_0^1 f_n(x)\,dx$ converges to 1 as $n \to \infty$.

38. Let $\{f_n\}_{n\geq1}$ be a sequence of continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ converging uniformly. Suppose
$$\lim_{n\to\infty} f_n(p, q) = 0,$$
for all rational numbers p and q. Prove that
$$\lim_{n\to\infty} f_n(x, y) = 0,$$
for all $(x, y) \in \mathbb{R}^2$.

39. Show that if $a > 0$, then the sequence of functions $f_n(x) = n^2x^2e^{-nx}$ converges uniformly on the interval $[a, \infty)$, but that it does not converge uniformly on the interval $[0, \infty)$.

40. Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)