Kannur University

Department of Mathematical Sciences

Real Analysis – Minor 2

August – December 2025

Maximum Marks: 24 Time: 90 minutes

Instructions: Answer all questions. Justify each answer clearly.

1. Determine whether each of the following statements is true or false. Justify your answer. (4 marks)

   (a) Every bounded infinite subset of $\mathbb{R}^n$ has a convergent subsequence.

   (b) Every countable subset of $\mathbb{R}^n$ is closed.

   (c) A complex number α is called an *algebraic number* if there exists a nonzero polynomial with rational coefficients

   $$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{Q},\ a_n \neq 0,$$

   such that

   $$p(\alpha) = 0.$$

   The set of all algebraic numbers in $\mathbb{R}$ is uncountable.

2. Let $S \subseteq \mathbb{R}^n$. Decide whether the following statements are true or false, with justification: (4 marks)

   (a) If S is countable, then $\mathbb{R}^n \setminus S$ is uncountable.

   (b) A countable union of countable sets in $\mathbb{R}^n$ is countable.

   (c) The set of rational points $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

3. A subset A of a metric space (X, d) is *bounded* if there exists a point $x_0 \in X$ and a real number $M > 0$ such that

   $$d(x, x_0) < M \quad \text{for all } x \in A.$$

   Using this definition, decide whether the following statements are true or false, with justification: (4 marks)

   (a) Every finite subset of a metric space is bounded.

   (b) In $(\mathbb{R}, d)$ with the usual metric, the set $[0, 1)$ is bounded but not compact.

   (c) In the discrete metric on an infinite set X, every subset is bounded.

4. In a metric space (X, d), a set $K \subset X$ is *compact* if every open cover of K has a finite subcover. Decide whether the following statement is true or false, with justification: Every closed and bounded set is compact. (3 marks)

5. Prove that the Cantor set $C \subset [0,1]$ contains no nonempty open interval. (3 marks)

6. A set $P \subset \mathbb{R}$ is *perfect* if it is closed and every point of P is a limit point of P. Decide whether the following statements are true or false, with justification: (2 marks)

   (a) The set $\mathbb{Q} \cap [0,1]$ is perfect.

   (b) The set $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ is perfect.

7. A subset E of a metric space (X,d) is *connected* if there do not exist disjoint open sets $U, V \subseteq X$ such that

$$E \subseteq U \cup V, \quad E \cap U \neq \varnothing, \quad E \cap V \neq \varnothing.$$

   Using this definition, decide whether the following statements are true or false, with justification: (5 marks)

   (a) The union of two connected subsets of a metric space with nonempty intersection is connected.

   (b) In a discrete metric space, the only connected subsets are singletons.

   (c) Every connected subset of $\mathbb{R}$ is an interval.

8. **Bonus Question:** Let d be a metric on X. Show that d and $d/(1+d)$ generate the same topology. Hint: compare open balls. (5 marks, optional)

   Rule for bonus: If the bonus is correct, it plus any 5 of the first 7 questions will count for 24 marks; you may skip 2 questions.