Kannur University

Department of Mathematical Sciences

Real Analysis - Minor 7

MSc 2025 Aug - Dec

Maximum Marks: To Be Determined By The Curve **Time: 60 Minutes**
Date: November 25, 2025

Instructions

Think about examples to get an idea about the result, whether it is true or not, or how it works. Take your time, think deep, conclude in the end. Use pen and paper and get your hands dirty.

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1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies the generalized Lipschitz condition:

$$|f(x) - f(y)| \leq |x - y|^\beta$$

for all $x, y \in \mathbb{R}$.

Which of the following statements is **correct**?

(1) If $\beta = 1$, then f is **differentiable**.

(2) If $\beta > 0$, then f is **uniformly continuous**.

(3) If $\beta > 1$, then f is a **constant function**.

(4) f must be a **polynomial**.

Hint: Think about **possible functions** that satisfy this condition, but **do not neglect the obvious ones!** You will need to know the rigorous definitions of **uniform continuity** and the standard method to prove a function is **constant**. Still seeking hints? Realize that the information already provided is, in fact, almost the solution.

2. Let $I = [1, 2] \cup [5, 8] \subset \mathbb{R}$. For $x \in \mathbb{R}$, define $f(x)$ as the **distance** from x to the set I:

$$f(x) = \inf\{|x - y| : y \in I\}.$$

Then:

(1) f is **discontinuous** somewhere on $\mathbb{R}$.

(2) f is **continuous** on $\mathbb{R}$, but **not differentiable** only at $x = 1$.

(3) f is **continuous** on $\mathbb{R}$, but **not differentiable** only at $x = 1$ and $x = 2$.

(4) f is **continuous** on $\mathbb{R}$, but **not differentiable** only at $x = 1, 3/2, 2$.

Hint: You do not need to prove this systematically. **Draw a picture** of the function $f(x)$ and the set I on the real line to demonstrate your idea and quickly determine the correct or incorrect options.

3. Suppose f is continuous for $x \geq 0$, differentiable for $x > 0$, $f(0) = 0$, and f' is **monotonically increasing**.

Define the function g by $g(x) = \frac{f(x)}{x}$ for $x > 0$.

Prove that g is **monotonically increasing**.

Hint: To prove that g is monotonically increasing, you must show that $g'(x) \geq 0$. Make sure to utilize all given assumptions (especially $f(0) = 0$ and f' being increasing). **Consider applying the Mean Value Theorem to f on the interval $[0, x]$.**

4. Examine the **uniform convergence** of the series of functions $\sum_{n=1}^{\infty} f_n(x)$, where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f_n(x) = \frac{(-1)^n}{n}$$

Hint: Think carefully about the definitions of a **series of functions**, its **pointwise convergence**, and **uniform convergence**. How does the fact that $f_n(x)$ does not depend on x simplify the relationship between these concepts?

5. Demonstrate, using distinct and rigorous examples, the difference in behavior between pointwise and uniform convergence regarding the interchange of the limit and integration operators.

Specifically, provide:

(a) An example of a sequence of functions $\{f_n\}$ on an interval $[a, b]$ that converges **pointwise** to f, where the limit and integral **cannot be interchanged**:

$$\lim_{n\to\infty} \left(\int_a^b f_n(x)\, dx \right) \neq \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx$$

Hint: Think about a sequence of functions whose **area (integral)** is non-decreasing, even though the sequence converges pointwise to zero.

(b) An example of a sequence of functions $\{g_n\}$ that converges **uniformly** to g, where the limit and integral **can be interchanged**:

$$\lim_{n\to\infty} \left(\int_c^d g_n(x)\, dx \right) = \int_c^d \left(\lim_{n\to\infty} g_n(x) \right) dx$$

6. Suppose that $\{f_n\}$ is a sequence of real-valued functions on $\mathbb{R}$. Suppose it converges to a **continuous function f uniformly on each closed and bounded subset of $\mathbb{R}$** (i.e., on every compact set).

Which of the following statements are **true**?

(1) The sequence $\{f_n\}$ converges to f **uniformly on $\mathbb{R}$**.

(2) The sequence $\{f_n\}$ converges to f **pointwise on $\mathbb{R}$**.

(3) For all sufficiently large n, the function f_n is **bounded**.

(4) For all sufficiently large n, the function f_n is **continuous**.

Hint: To determine the truth of these statements, **construct illustrative counterexamples**, either from familiar functions or by creating new ones. Specifically, consider the following lines of inquiry:

- Can you find a sequence of functions that converges **uniformly** on $\mathbb{R}$ to an **unbounded function**?
- Can you construct a sequence of functions that are **discontinuous** on $\mathbb{R}$ but converge **uniformly to zero** on $\mathbb{R}$?
- Analyze sequences of functions defined piecewise using the structure of increasingly large compact intervals, such as $[-n, n]$.

"Our lives are defined by opportunities, even the ones we miss." - Benjamin Button