# Tutorial Sheet 2- Real Analysis

23 November 2025

## 1 Differentiation and Sequence of Functions

1. Say that $f$ is differentiable and $f'(x) \neq 1$ on $(-\infty, \infty)$. Show that there is at most one real number $a$ such that $f(a) = a$. *Hint:* Suppose to the contrary there are two distinct values $a$ and $b$ such that $f(a) = a$ and $f(b) = b$. Define $g(x) = f(x) - x$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f'(x^2) = 4x^2 - 1$ for $x > 0$ and $f(1) = 1$. Find the value of $f(4)$. *Hint:* Let $t = x^2$. Then integrate $f'(t)$. Use the value of $f(1)$. Calculate $f(4)$.

3. Let $a < c < b$. Suppose $f : (a, b) \to \mathbb{R}$ is continuous on $(a, b)$ and differentiable at every point of $(a, b) \setminus \{c\}$. If $\lim_{x\to c} f'(x)$ exists, prove that $f$ is differentiable at $c$ and that $f'(c) = \lim_{x\to c} f'(x)$. *Hint:* By the Mean Value Theorem, there is $k_x \in (x, c)$ s.t. $\frac{f(c)-f(x)}{c-x} = f'(k_x)$. Use this to evaluate the limit of the difference quotient.

4. Discuss the differentiability of the function $f(x) = |x|^3$ at $x = 0$. *Hint:* Use the definition of the derivative at a point, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x}$. Consider the left-hand limit and the right-hand limit separately.

5. Let $f, f_n : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \log x$ and $f_n(x) = \log\left(x + \frac{1}{n}\right)$. Discuss the convergence of the sequence of functions $(f_n)$ to $f$. *Hint:* Consider the pointwise limit $\lim_{n\to\infty} f_n(x)$. For uniform convergence, evaluate the difference $|f_n(x) - f(x)| = \log\left(1 + \frac{1}{nx}\right)$. Examine the behavior of the supremum of this difference, perhaps by evaluating $|f_n(x) - f(x)|$ at $x = 1/n^2$.