Riemann Integration

A Comprehensive Revision Worksheet

150 problems | Undergraduate foundations and postgraduate reasoning

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Instructions. This is a ten-day revision exercise, not a day-by-day assignment. Attempt topics in any order. Justify all answers; for false statements provide counterexamples. Verify the hypotheses of every theorem you invoke. Hints for selected questions appear at the end and should be consulted only after a serious attempt. Unless stated otherwise, intervals are nondegenerate compact real intervals, and "integrable" means properly Riemann integrable.

Scope. Sections 1–6 and 10 emphasize the classical Riemann theory; Sections 7–9 add advanced connections (null sets, convergence of functions, and improper integration). These extensions are marked by their section headings so that students may choose their own order.

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1 Riemann integrability and Darboux sums

1.1. State the definitions of a partition, upper sum, lower sum, upper integral, and lower integral of a bounded function on $[a, b]$.

1.2. Prove that $L(f, P) \leq U(f, P)$ for every partition P, and that refining a partition increases the lower sum and decreases the upper sum.

1.3. Prove that for any two partitions P, Q, one has $L(f, P) \le U(f, Q)$.

1.4. State and prove the Darboux criterion: f is Riemann integrable if and only if for every $\varepsilon > 0$ there is a partition P with $U(f, P) - L(f, P) < \varepsilon$.

1.5. Compute upper and lower sums of $f(x) = x$ on $[0, 1]$ for the uniform partition into n equal pieces; deduce its integral.

1.6. Repeat the preceding calculation for $f(x) = x^2$ on $[0, 1]$, using $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.

1.7. Show directly from Darboux sums that a constant function $f(x) = c$ is integrable and find its integral.

1.8. Let $f(x) = 1$ for $x \in [0, 1/2)$ and $f(x) = 3$ for $x \in [1/2, 1]$. Prove integrability and compute the integral.

1.9. Let $f(x) = 1$ on the rationals and 0 on the irrationals in $[0, 1]$. Compute every upper and lower sum. What follows?

1.10. True or false (prove or disprove): if a bounded function has one partition P with $U(f, P) - L(f, P) = 0$, then f is constant.

1.11. Show that changing the value of an integrable function at one point preserves integrability and does not change its integral.

1.12. Prove that if f is integrable on $[a, b]$, then $|f|$ is integrable. Is the converse true?

1.13. Find an example of a bounded function on $[0, 1]$ whose upper integral is 1 and lower integral is 0.

1.14. Suppose f is bounded and $U(f, P_n) - L(f, P_n) \to 0$ for some sequence of partitions (P_n). Prove that f is integrable.

1.15. Explain why boundedness is included in the definition of the (proper) Riemann integral. Give an unbounded function with a finite improper integral.

2 Integrable classes: continuity, monotonicity, and discontinuities

2.1. Prove that every continuous function on a compact interval is Riemann integrable, using uniform continuity.

2.2. Prove that every bounded monotone function on $[a, b]$ is Riemann integrable.

2.3. Give an integrable function that is discontinuous at exactly one point, and justify integrability directly.

2.4. Construct an integrable function on $[0, 1]$ with infinitely many discontinuities.

2.5. Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x on $[0, 1]$. Determine whether f is Riemann integrable.

2.6. Define $f(0) = 0$ and $f(x) = 1/n$ when $x = 1/n$ $(n \ge 1)$, and $f(x) = 0$ otherwise. Prove

integrability and find its integral.

2.7. Prove that a bounded function with finitely many discontinuities on $[a, b]$ is Riemann integrable.

2.8. Does the converse hold: must a Riemann-integrable function have only finitely many discontinuities? Justify.

2.9. Give an example of a function discontinuous at every rational point and continuous at every irrational point of $[0, 1]$; discuss its Riemann integrability.

2.10. Prove that a piecewise continuous bounded function with finitely many pieces is Riemann integrable.

2.11. If f is Riemann integrable and $g = f$ except at finitely many points, prove that g is Riemann integrable with the same integral.

2.12. True or false: a bounded function with a dense set of discontinuities cannot be Riemann integrable. Justify carefully.

2.13. Prove that if a bounded function is continuous except on a finite set, then the oscillation sums can be made arbitrarily small.

2.14. Find a bounded function that is continuous on $(0, 1]$ but not Riemann integrable on $[0, 1]$, or prove that no such example exists.

2.15. Let $f(x) = \sin(1/x)$ for $x > 0$ and $f(0) = 0$. Prove that f is Riemann integrable on $[0, 1]$.

3 Algebraic properties and inequalities

3.1. Prove the linearity of the Riemann integral for integrable functions using upper and lower sums or the integrability criterion.

3.2. If $f \le g$ on $[a, b]$ and both are integrable, prove $\int_a^b f \le \int_a^b g$.

3.3. Prove $\left|\int_a^b f\right| \le \int_a^b |f|$ for every Riemann-integrable f.

3.4. Show that the product of two Riemann-integrable functions is Riemann integrable.

3.5. Prove that if f is integrable and $|f(x)| \le M$, then $\left|\int_a^b f\right| \le M(b - a)$.

3.6. If f and g are integrable, prove that $\max(f, g)$ and $\min(f, g)$ are integrable.

3.7. Show that if f is integrable and $f(x) \ge 0$, then $\int_a^b f \ge 0$. Can equality hold for a nonzero function?

3.8. Suppose f is continuous, nonnegative on $[a, b]$, and $\int_a^b f = 0$. Prove that $f \equiv 0$.

3.9. Give a nonnegative Riemann-integrable function with integral zero that is not identically zero.

3.10. If f is integrable and $|f(x)| \ge c > 0$ for every x, must $1/f$ be integrable? Prove or disprove.

3.11. Prove that if f is integrable and ϕ is Lipschitz on an interval containing the range of f, then $\phi \circ f$ is integrable.

3.12. Deduce from the preceding result that f^2 is integrable whenever f is bounded and integrable.

3.13. Prove that $\int_a^b fg$ satisfies $|\int_a^b fg| \le (\int_a^b f^2)^{1/2}(\int_a^b g^2)^{1/2}$ for integrable f, g.

3.14. True or false: $\int_a^b f = \int_a^b g$ implies $f = g$ at some point. Consider separately continuous and merely integrable functions.

3.15. For Riemann-integrable functions f and g, derive the inequality $|\int_a^b f - \int_a^b g| \le \int_a^b |f - g|$ and explain its use.

4 Intervals, additivity, and integral as a function of the endpoint

4.1. Prove that f is integrable on $[a, b]$ if and only if it is integrable on both $[a, c]$ and $[c, b]$, for $a < c < b$.

4.2. Establish $\int_a^b f = \int_a^c f + \int_c^b f$ for integrable f.

4.3. For integrable f, define $F(x) = \int_a^x f(t)\,dt$. Prove that F is Lipschitz on $[a, b]$.

4.4. Show that $F(x) = \int_a^x f(t)\,dt$ is continuous even if f is discontinuous.

4.5. Let f be the step function $f = 1$ on $[0, 1/2)$ and $f = 2$ on $[1/2, 1]$. Find $F(x) = \int_0^x f(t)\,dt$ and determine where F is differentiable.

4.6. Prove that if f is Riemann integrable on $[a, b]$ and continuous at $c \in (a, b)$, then $F'(c) = f(c)$ for $F(x) = \int_a^x f(t)\,dt$.

4.7. Can $F'(c)$ exist when f is discontinuous at c? Construct an example.

4.8. If f is nonnegative and integrable, prove that $F(x) = \int_a^x f$ is nondecreasing.

4.9. If f is integrable and $F(x) = \int_a^x f$, can F be constant while f is nonzero? Give a rigorous example or proof.

4.10. Suppose f is continuous on $[a, b]$. Prove that there exists $c \in [a, b]$ with $\int_a^b f = f(c)(b - a)$.

4.11. Show that the preceding mean-value conclusion may fail for an integrable function that is not continuous.

4.12. Let f be integrable on $[0, 1]$. Prove $\lim_{h \downarrow 0} \int_0^h f(x)\,dx = 0$.

4.13. If f is integrable on $[a, b]$, show that $\int_a^b f$ can be approximated by finite sums over any sequence of partitions with mesh tending to zero, provided tags are chosen arbitrarily.

4.14. Determine whether the conclusion of the preceding question remains valid if mesh tending to zero is omitted.

4.15. If f is continuous and $\int_a^x f(t)\,dt = 0$ for every $x \in [a, b]$, prove $f \equiv 0$.

5 Riemann sums and equivalent formulations

5.1. Define a tagged partition and its Riemann sum. State the tagged-partition criterion for Riemann integrability.

5.2. Prove that an integrable bounded function has Riemann sums converging to its integral as the mesh tends to zero, uniformly over choices of tags.

5.3. For bounded f, show that if, for every $\varepsilon > 0$, all sufficiently fine tagged partitions have their Riemann sums within ε of I, then the Darboux upper and lower integrals both equal I.

5.4. Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}(k/n)^3$ by identifying a Riemann sum.

5.5. Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sqrt{1+(k/n)^2}$ as a definite integral; an unevaluated integral is acceptable.

5.6. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$.

5.7. Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\log(1+k/n)$.

5.8. Show that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(k/n) = \int_0^1 f(x)\,dx$ whenever f is Riemann integrable.

5.9. Explain why the preceding formula can fail if f is bounded but not Riemann integrable; provide an example.

5.10. Prove that for continuous f, left-endpoint and right-endpoint sums along uniform partitions have the same limit.

5.11. For an increasing f, estimate the difference between upper and lower sums on the uniform partition in terms of $f(b) - f(a)$.

5.12. Prove that $\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+k/n} \to \log 2$.

5.13. Find the limit of $\sum_{k=1}^{n} \frac{k}{n^2+k^2}$ and justify every normalization step.

5.14. Let f be continuous on $[0,1]$. Prove $\frac{1}{n}\sum_{k=1}^{n}(-1)^k f(k/n) \to 0$.

5.15. Give an example showing that convergence of Riemann sums along one specially chosen sequence of tagged partitions does not imply integrability.

6 Fundamental theorem of calculus

6.1. State both parts of the fundamental theorem of calculus, including their hypotheses.

6.2. If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, prove $F'(x) = f(x)$ for $x \in (a,b)$.

6.3. If F is differentiable on $[a,b]$ with Riemann-integrable derivative, prove $\int_a^b F'(x)\,dx = F(b) - F(a)$.

6.4. Explain why the mere existence of F' does not guarantee that F' is Riemann integrable.

6.5. Evaluate $\int_0^1 3x^2\,dx$ using an antiderivative, and compare with a Darboux-sum calculation.

6.6. Let $F(x) = \int_0^{x^2} \cos(t^2)\,dt$. Compute $F'(x)$.

6.7. For $x > 0$, let $F(x) = \int_x^{2x} \frac{dt}{1+t^2}$. Compute $F'(x)$.

6.8. Suppose f is continuous and $G(x) = \int_{u(x)}^{v(x)} f(t)\,dt$ with differentiable u, v. Derive a formula for $G'(x)$.

6.9. If f is continuous and $\int_a^x f(t)\,dt = x^2 - a^2$ for all $x \in [a, b]$, determine f.

6.10. Show that if f is continuous and $\int_a^b f = 0$, there exists $c \in [a, b]$ with $f(c) = 0$.

6.11. Give a Riemann-integrable f for which $F(x) = \int_a^x f$ is not differentiable at some interior point.

6.12. Prove integration by parts for continuously differentiable functions, using the fundamental theorem.

6.13. Prove the substitution rule for a continuously differentiable monotone change of variable, clearly stating the assumptions.

6.14. Evaluate $\int_0^1 \frac{2x}{1+x^2}\,dx$ by substitution.

6.15. A student writes: "Every integrable function is the derivative of its indefinite integral." Identify the error and give a corrected statement.

7 Oscillation, null sets, and Lebesgue's criterion

7.1. Define the oscillation of a bounded function on an interval and explain its relation to upper minus lower Darboux sums.

7.2. Define a set of Lebesgue measure zero using countable coverings by intervals of arbitrarily small total length.

7.3. State Lebesgue's criterion for Riemann integrability of a bounded function on $[a, b]$.

7.4. Use Lebesgue's criterion to show that a bounded function with countably many discontinuities is Riemann integrable.

7.5. Explain why the statement "every countable set has measure zero" does not imply that the Dirichlet function is Riemann integrable.

7.6. Determine the discontinuity set of Thomae's function $f(x) = 1/q$ when $x = p/q$ in lowest terms and $f(x) = 0$ for irrational x; prove integrability.

7.7. Construct a Riemann-integrable function whose discontinuity set is dense in $[0, 1]$.

7.8. Prove that a finite union of null sets is null. Does the same hold for countable unions? Justify.

7.9. Show that a bounded function whose discontinuity set is a finite set is integrable, using Lebesgue's criterion.

7.10. Explain why an uncountable discontinuity set does not automatically prevent Riemann

integrability.

7.11. Let C be the middle-thirds Cantor set and $f = \mathbf{1}_C$. Assuming C is null and closed with empty interior, prove f is Riemann integrable.

7.12. Find the value of $\int_0^1 \mathbf{1}_C(x)\,dx$ under the hypotheses of the preceding question.

7.13. Prove that if a bounded function is continuous almost everywhere on $[a, b]$, then it is Riemann integrable.

7.14. True or false: two bounded functions equal almost everywhere are either both Riemann integrable or both not Riemann integrable. Justify.

7.15. Describe why the boundedness assumption cannot simply be dropped from Lebesgue's criterion for proper Riemann integration.

8 Sequences of functions and interchange of limits

8.1. State the definitions of pointwise and uniform convergence of functions on $[a, b]$.

8.2. Prove that a uniform limit of Riemann-integrable functions on $[a, b]$ is Riemann integrable.

8.3. If $f_n \to f$ uniformly and each f_n is Riemann integrable, prove $\int_a^b f_n \to \int_a^b f$.

8.4. Construct continuous functions f_n on $[0, 1]$ converging pointwise to a discontinuous function.

8.5. Give a sequence of continuous functions converging pointwise to a Riemann-integrable limit but whose integrals do not converge to the integral of the limit.

8.6. Consider $f_n(x) = x^n$ on $[0, 1]$. Find the pointwise limit, test uniform convergence, and compare integrals.

8.7. Consider $f_n(x) = nx(1 - x)^n$ on $[0, 1]$. Determine the pointwise limit and the limit of the integrals.

8.8. Suppose each f_n is Riemann integrable and $\|f_n - f\|_\infty \to 0$. Show that $|\int_a^b f_n - \int_a^b f| \le (b - a)\|f_n - f\|_\infty$.

8.9. True or false: pointwise convergence of integrable functions implies integrability of the limit. Construct a counterexample if false.

8.10. Can a sequence of step functions converge uniformly to every continuous function on $[a, b]$? Prove or disprove.

8.11. Prove that every continuous function on $[a, b]$ can be uniformly approximated by step functions.

8.12. If $f_n \ge 0$ and $f_n \to 0$ pointwise, must $\int_a^b f_n \to 0$? Give an example or a proof.

8.13. Suppose f_n are continuous, $f_n \to f$ uniformly, and f is continuous. Does $\int_a^b f_n \to \int_a^b f$ follow without any extra condition?

8.14. Show that if f_n are integrable and $\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$, then $\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n$.

8.15. A student claims that boundedness of every f_n and pointwise convergence suffice to interchange limit and integral. Explain the logical gap.

9 Improper integrals and convergence tests

9.1. Define $\int_a^\infty f(x)\,dx$ as an improper Riemann integral and distinguish it from a proper integral.

9.2. Determine for which $p > 0$ the improper integral $\int_1^\infty x^{-p}\,dx$ converges.

9.3. Determine for which $p > 0$ the improper integral $\int_0^1 x^{-p}\,dx$ converges.

9.4. Prove the comparison test for improper integrals of nonnegative functions.

9.5. Prove a Cauchy criterion for convergence of $\int_a^\infty f(x)\,dx$.

9.6. Determine whether $\int_1^\infty \frac{\sin x}{x}\,dx$ converges, and whether it converges absolutely.

9.7. Show that $\int_1^\infty \frac{|\sin x|}{x}\,dx$ diverges.

9.8. Determine the convergence of $\int_2^\infty \frac{dx}{x(\log x)^p}$ as a function of p.

9.9. Investigate $\int_0^1 \frac{dx}{x(\log(1/x))^2}$; identify every problematic endpoint before drawing a conclusion.

9.10. Explain why $\int_{-1}^1 \frac{1}{x}\,dx$ does not exist as an improper integral although its Cauchy principal value is zero.

9.11. Prove that if $f \geq 0$ is continuous on $[a, \infty)$ and $\int_a^\infty f < \infty$, it need not follow that $f(x) \to 0$. Construct a continuous example.

9.12. Show that if f is nonnegative, decreasing, and $\int_1^\infty f(x)\,dx < \infty$, then $xf(x) \to 0$.

9.13. Establish the integral test for a positive decreasing function f and the series $\sum_{n=1}^\infty f(n)$.

9.14. For which real p does $\int_1^\infty x^p e^{-x}\,dx$ converge? Justify without evaluating it.

9.15. Give an example of an improper integral that converges conditionally but not absolutely, and explain why this is analogous to numerical series.

10 Synthesis, counterexamples, and proof diagnosis

10.1. Is every derivative Riemann integrable? Give a precise statement with proof or counterexample.

10.2. Can a function on $[0, 1]$ be Riemann integrable on every closed subinterval of $(0, 1)$ but fail to be Riemann integrable on $[0, 1]$? Give an example.

10.3. If f is bounded and $\int_a^b f = 0$, must f vanish somewhere? Analyze the continuous and integrable cases.

10.4. Find a bounded Riemann-integrable f such that $f^2 = f$ but f is not continuous.

10.5. Prove that a Riemann-integrable function on $[a, b]$ can be approximated in the integral sense by step functions.

10.6. If f is Riemann integrable and g is continuous, prove $g \circ f$ is Riemann integrable.

10.7. If f is Riemann integrable and g is merely Riemann integrable, is $g \circ f$ necessarily Riemann integrable? Construct a counterexample.

10.8. Prove that if f is continuous and $\int_a^b f(x)g(x)\,dx = 0$ for every continuous g, then $f \equiv 0$.

10.9. Suppose f is integrable and $\int_a^b f(x)g(x)\,dx = 0$ for every continuous g. What can be concluded about f? Justify.

10.10. Construct two Riemann-integrable functions with the same integral but different discontinuity sets.

10.11. A student argues: "f is bounded, so $U(f, P) - L(f, P)$ becomes small when the partition is fine." Explain why this is false.

10.12. A student argues: "The integral is zero because the function is zero on a dense set." Supply a counterexample or conditions that repair the claim.

10.13. Prove that if f is continuous on $[a, b]$ and $\int_a^b f(x)^2\,dx = 0$, then $f \equiv 0$. What changes if f is only integrable?

10.14. Show that $\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\,dx = 0$, and explain why uniform convergence is not the relevant argument.

10.15. Compare the Darboux criterion, the tagged Riemann-sum criterion, and Lebesgue's criterion: state what each tests and outline the logical relationships between them.

Selected hints

Hints are deliberately brief and do not constitute solutions. Problems not listed here are intended to be attempted without hints.

Section 1: Riemann integrability and Darboux sums

1.4 Choose a partition that makes the total oscillation contribution small; use the definitions of upper and lower integrals.

1.6 Compare right and left endpoint sums.

1.10 A zero gap forces each subinterval's oscillation to vanish.

1.12 Consider a function with different values on two dense sets.

1.15 Separate proper from improper integration.

Section 2: Integrable classes: continuity, monotonicity, and discontinuities

2.1 Use uniform continuity to control oscillations on sufficiently short intervals.

2.2 For a monotone function, the sum of oscillations telescopes.

2.5 Every nondegenerate interval contains rational and irrational points.

2.6 Only finitely many spikes exceed a prescribed positive threshold.

2.9 Think of the denominator of a rational number in lowest terms.

2.12 A dense set can still have measure zero.

2.14 Consider how continuity away from one endpoint restricts discontinuities.

Section 3: Algebraic properties and inequalities

3.4 First establish integrability of the square of an integrable function.

3.6 Use $\max(f, g) = (f + g + |f - g|)/2$.

3.10 The map $t \mapsto 1/t$ is Lipschitz on $(-\infty, -c] \cup [c, \infty)$ separately; use a direct difference estimate.

3.11 Control oscillations by the Lipschitz constant.

3.13 Apply $\int (f - tg)^2 \geq 0$ and inspect the discriminant.

3.14 The intermediate value property is available only in the continuous case.

Section 4: Intervals, additivity, and integral as a function of the endpoint

4.3 Bound $|F(y) - F(x)|$ by $\|f\|_\infty |y - x|$.

4.6 Write the difference quotient as an average of f near c.

4.7 Try changing f at one point.

4.9 A function can be nonzero on a null set.

4.11 Use a step function whose average is not an attained value.

4.13 Compare arbitrary tagged sums with upper and lower sums.

Section 5: Riemann sums and equivalent formulations

5.2 Place each tagged sum between the Darboux sums of its underlying partition.

5.3 Choose tags nearly attaining the supremum and infimum on each subinterval.

5.6 Rewrite as $\frac{1}{n}\sum 1/(1+(k/n)^2)$.

5.9 The Dirichlet function has right-endpoint sums equal to one.

5.14 Pair consecutive terms and use uniform continuity.

5.15 A fixed choice of rational tags hides the behavior of the Dirichlet function.

Section 6: Fundamental theorem of calculus

6.3 Apply the mean value theorem on each small subinterval.

6.4 A derivative can be unbounded even on a compact interval.

6.7 Differentiate the two variable endpoints.

6.10 A continuous function with no zeros has constant sign.

6.11 Use a jump discontinuity.

6.13 Apply the chain rule to an antiderivative.

6.15 Continuity of the integrand at a point is sufficient for the integral function to have the expected derivative there; consider what happens at a jump discontinuity.

Section 7: Oscillation, null sets, and Lebesgue's criterion

7.4 Density of discontinuities is not the same as density of points where the function is nonzero.

7.6 For each $\varepsilon > 0$, only finitely many rational points have value at least ε.

7.10 The Cantor set is an uncountable null set.

7.14 Change the zero function on all rational points.

Section 8: Sequences of functions and interchange of limits

8.2 First show that f is bounded. Then use $U(f,P) - L(f,P) \le U(f_n,P) - L(f_n,P) + 2(b-a)\|f_n - f\|_\infty$.

8.5 Try a tall triangular spike whose width shrinks.

8.6 The limit differs at $x = 1$.

8.9 Approximate the Dirichlet function by indicators of finite sets.

8.10 Use uniform continuity and piecewise constant approximations.

8.12 Again use a moving spike with constant area.

8.14 Uniform convergence of the partial sums follows from the Weierstrass M-test.

Section 9: Improper integrals and convergence tests

9.5 Apply the Cauchy condition to differences of truncated integrals: control $|\int_R^S f(x)\,dx|$ for all sufficiently large R, S.

9.6 Integrate by parts or apply Dirichlet's test.

9.7 On each period, $|\sin x|$ has a fixed positive integral.

9.9 Check both $x \downarrow 0$ and $x \uparrow 1$.

9.11 Construct narrow, increasingly tall, disjoint triangular spikes.

9.12 Compare $xf(x)$ with the integral over $[x/2, x]$.

9.13 Compare rectangles with areas under the graph.

Section 10: Synthesis, counterexamples, and proof diagnosis

10.1 Try a differentiable function whose derivative is unbounded.

10.3 The intermediate value theorem gives more in the continuous case.

10.6 Use uniform continuity of g on a compact interval containing the bounded range of f.

10.7 Let f be the distance to a closed nowhere-dense set of positive measure, and let $g = \mathbf{1}_{\{0\}}$.

10.8 Choose $g = f$.

10.9 First prove the integral of f^2 is zero by approximation of f with continuous functions, or use positivity and continuity test functions.

10.11
Consider the Dirichlet function.

10.12
Without integrability, consider the Dirichlet function. If f is Riemann integrable and zero on a dense set, choose a zero-valued tag in every subinterval.

10.14
Find an antiderivative or bound the integral by $\log(1 + n^2)/(2n)$.

10.15
The criteria are equivalent for bounded functions, but they use different descriptions of integrability.