Numerical Sequences and Series

A Comprehensive Revision Worksheet

150 problems | Undergraduate foundations and postgraduate reasoning

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Instructions. This is a ten-day revision exercise, not a day-by-day assignment. Attempt topics in any order. Give a proof or counterexample for every true/false assertion; justify all multiple-choice answers. Verify hypotheses whenever applying a theorem. Consult the section-wise hints at the end only after a serious attempt. Unless stated otherwise, sequences and series are real-valued, and indices begin at $n = 1$.

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1 Sequences and convergence

1.1. State the ε–N definition of $x_n \to L$. Prove directly that $(3n-2)/(5n+1) \to 3/5$.

1.2. Prove that a convergent sequence has a unique limit. Identify the precise step at which the Archimedean property is used, if any.

1.3. Prove that every convergent real sequence is bounded. Give an unbounded sequence with a bounded subsequence.

1.4. Determine, with justification, whether $x_n = (-1)^n + n^{-1}$ converges. Does $|x_n|$ converge?

1.5. Show from the definition that $n^{-p} \to 0$ for every $p > 0$. You may use the Archimedean property.

1.6. True or false (justify each): (a) changing finitely many terms changes the limit; (b) a bounded sequence must converge; (c) if $x_n \to 0$, then $x_n^2 \to 0$.

1.7. Construct a sequence taking infinitely many distinct values that converges to 0, and another taking only two values that does not converge.

1.8. Suppose $x_n \to L$ and $x_n \geq c$ for all sufficiently large n. Prove $L \geq c$. Is the conclusion $L > c$ valid?

1.9. Prove that if $x_n \to L > 0$, then $x_n > 0$ eventually. Show why the conclusion can fail when $L = 0$.

1.10. Let $x_n = \sqrt{n^2 + 3n} - n$. Find its limit without appealing to an unproved "dominant terms" rule.

1.11. Decide whether $x_n = \sin(n\pi/2) + 1/n$ converges; determine all values attained infinitely often by the trigonometric part.

1.12. Suppose $|x_n - L| \leq 2^{-n}$ for all n. Find an explicit $N(\varepsilon)$ that proves $x_n \to L$.

1.13. Give an example of a sequence (x_n) such that $x_{2n} \to 0$ but (x_n) does not converge. What additional condition on (x_{2n+1}) ensures convergence?

1.14. Prove that $x_n \to L$ if and only if every neighborhood of L contains all but finitely many terms of the sequence.

1.15. Let $x_n \in \{0, 1\}$ for all n. Prove that (x_n) converges if and only if it is eventually constant.

2 Limit theorems and inequalities

2.1. Prove the sum and product limit laws directly from the definition of convergence.

2.2. If $x_n \to a$ and $y_n \to b \neq 0$, prove $x_n/y_n \to a/b$ and explain why $y_n \neq 0$ eventually.

2.3. Prove the squeeze theorem. Apply it to $x_n = \sin(n^2)/\sqrt{n}$.

2.4. If $x_n \to a$, show $|x_n| \to |a|$. Does the converse hold? Supply a counterexample.

2.5. Suppose $x_n \leq y_n \leq z_n$ eventually and $x_n, z_n \to L$. Prove $y_n \to L$.

2.6. Evaluate $\lim_{n\to\infty}(\sqrt{n^2 + an + b} - n)$ for fixed $a, b \in \mathbb{R}$, when the square root is defined eventually.

2.7. Show that if $x_n \to 0$ and (y_n) is bounded, then $x_n y_n \to 0$. Is boundedness necessary?

2.8. True or false: if $x_n y_n \to 0$, then $x_n \to 0$ or $y_n \to 0$. Justify.

2.9. Suppose $x_n \to a$ and $x_n^2 \to b$. Prove $b = a^2$. Is the converse implication $x_n^2 \to a^2 \Rightarrow x_n \to a$

valid?

2.10. If $x_n \to a$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous at a, prove $f(x_n) \to f(a)$ using the definition of continuity.

2.11. Find the limit of $\frac{2n^3-n+1}{7n^3+4n^2-3}$, explicitly identifying the limit laws used.

2.12. Assume $x_n \geq 0$ and $x_n \to a$. Prove $\sqrt{x_n} \to \sqrt{a}$, treating $a = 0$ separately.

2.13. Suppose $x_n \to a$ and $y_n - x_n \to 0$. Prove $y_n \to a$. Is the statement still true if only $|y_n - x_n| \leq 1$?

2.14. Show that $\lim_{n\to\infty} \frac{n^p}{c^n} = 0$ for $p > 0$ and $c > 1$. You may first prove it for integer exponents and compare.

2.15. Find sequences (x_n) and (y_n) such that neither converges but $x_n + y_n$ and $x_n y_n$ both converge. Explain the phenomenon.

3 Monotone and recursively defined sequences

3.1. State and prove the monotone convergence theorem for increasing bounded sequences using the least-upper-bound property.

3.2. Give an increasing sequence that diverges to $+\infty$ and a strictly increasing sequence that converges.

3.3. Show that $x_n = 1 - 1/n$ is increasing and bounded; determine its supremum and limit.

3.4. Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$. Prove convergence and find the limit.

3.5. Let $x_1 > 0$ and $x_{n+1} = \frac{1}{2}(x_n + 2/x_n)$. Prove $x_n \to \sqrt{2}$. Treat the possibility $x_1 < \sqrt{2}$ carefully.

3.6. Prove that a monotone sequence converges if and only if it is bounded.

3.7. Define $x_1 = 0$ and $x_{n+1} = (x_n + 1)/2$. Obtain a formula for x_n and a second proof of convergence by monotonicity.

3.8. Suppose (x_n) is increasing and has a convergent subsequence. Prove that the full sequence converges to the same limit.

3.9. True or false: if $x_{n+1} - x_n \to 0$, then (x_n) converges. Give a counterexample if false.

3.10. Let $x_1 = 1$ and $x_{n+1} = 1/(2 + x_n)$. Show that (x_n) converges, although it is not monotone. Hint: investigate even and odd subsequences or a contraction estimate.

3.11. Suppose $0 \leq x_1 \leq 1$ and $x_{n+1} = x_n(2 - x_n)$. Determine the limit for every possible x_1.

3.12. Prove that if (x_n) is increasing and unbounded above, then $x_n \to +\infty$.

3.13. Construct a bounded sequence that is neither monotone nor eventually monotone, yet

converges.

3.14. Let $a, b > 0$, $x_1 = \sqrt{ab}$, and $x_{n+1} = (x_n + (a+b)/2)/2$. Determine its limit and prove convergence.

3.15. Let $x_1 \in (0,1)$ and $x_{n+1} = x_n^2$. Prove convergence and find the limit. What changes for $x_1 = 1$ or $x_1 > 1$?

4 Subsequences, Bolzano–Weierstrass, limsup and liminf

4.1. Define a subsequence precisely. Prove that every subsequence of a convergent sequence has the same limit.

4.2. Prove the Bolzano–Weierstrass theorem for bounded real sequences.

4.3. Find all subsequential limits of $x_n = (-1)^n + 1/n$.

4.4. Construct a bounded sequence whose set of subsequential limits is exactly $\{0, 1, 2\}$.

4.5. Show that if both (x_{2n}) and (x_{2n-1}) converge to L, then $x_n \to L$.

4.6. True or false: if a sequence has exactly one subsequential limit, it converges. State an additional hypothesis that makes the claim true.

4.7. Prove that every bounded sequence has a monotone subsequence.

4.8. Define $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ for bounded sequences using tails. Prove that

$$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n.$$

4.9. Compute the limsup and liminf of $x_n = (-1)^n(1 + 1/n)$.

4.10. Prove that a bounded sequence converges if and only if its limsup and liminf are equal.

4.11. Find the limsup and liminf of $x_n = \sin(n\pi/2) + 1/n$.

4.12. Prove that for a bounded sequence the limsup is a subsequential limit, and no subsequential limit exceeds it.

4.13. Construct a sequence whose set of subsequential limits is the whole interval $[0, 1]$; justify both inclusions.

4.14. Suppose $x_n \le y_n$ for every n, with both sequences bounded. Prove $\limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n$ and the corresponding liminf inequality.

4.15. Give a bounded sequence with infinitely many subsequential limits but no limit; explain why Bolzano–Weierstrass alone does not imply convergence.

5 Cauchy sequences, completeness and proper divergence

5.1. State the Cauchy criterion for real sequences. Prove that every convergent sequence is Cauchy.

5.2. Prove that every Cauchy sequence is bounded.

5.3. Prove that every Cauchy sequence in $\mathbb{R}$ converges, using Bolzano–Weierstrass.

5.4. Show directly that $x_n = \sum_{k=1}^{n} 2^{-k}$ is Cauchy, without first calculating its limit.

5.5. Decide whether the sequence of harmonic partial sums $H_n = \sum_{k=1}^{n} 1/k$ is Cauchy. Justify your answer.

5.6. Show that if $|x_{n+1} - x_n| \le 3^{-n}$ for all n, then (x_n) converges.

5.7. True or false: $|x_{n+1} - x_n| \to 0$ implies (x_n) is Cauchy. Justify.

5.8. Define $x_n \to +\infty$ and $x_n \to -\infty$ rigorously. Prove that $n^2 \to +\infty$.

5.9. Give a divergent sequence that diverges neither to $+\infty$ nor to $-\infty$.

5.10. Prove that if $x_n \to +\infty$ and $y_n \ge x_n$ eventually, then $y_n \to +\infty$.

5.11. If $x_n \to +\infty$ and $x_n > 0$ eventually, show $1/x_n \to 0$. Is the converse true under eventual positivity?

5.12. Construct a sequence that is not Cauchy but has a Cauchy subsequence.

5.13. Let (x_n) satisfy $|x_m - x_n| \le 1/n$ whenever $m \ge n$. Prove that it converges.

5.14. Explain why the sequence of rational decimal truncations of $\sqrt{2}$ is Cauchy in $\mathbb{Q}$ but has no limit in $\mathbb{Q}$. What property of $\mathbb{R}$ is missing from $\mathbb{Q}$?

5.15. Suppose (x_n) is Cauchy and $(x_{n_k}) \to L$. Give a direct proof that $x_n \to L$.

6 Infinite series and partial sums

6.1. Define convergence of $\sum_{n=1}^{\infty} a_n$ through partial sums. Prove that convergence implies $a_n \to 0$.

6.2. Show that $\sum_{n=1}^{\infty} 1/n$ diverges by grouping consecutive blocks of terms.

6.3. Determine the sum of $\sum_{n=1}^{\infty} 3/4^n$ and justify the passage to the infinite sum.

6.4. Determine whether $\sum_{n=1}^{\infty} 1/[n(n+1)]$ converges and find its sum.

6.5. Prove the Cauchy criterion for convergence of a numerical series.

6.6. Show that $\sum_{n=1}^{\infty} (-1)^n$ diverges even though its partial sums are bounded.

6.7. Prove that changing finitely many terms cannot affect convergence of a series. How can it affect the sum?

6.8. If $\sum a_n$ and $\sum b_n$ converge, prove $\sum(\alpha a_n + \beta b_n)$ converges and identify its sum.

6.9. Give a divergent series whose terms tend to zero, and a divergent series whose terms do not tend to zero.

6.10. Find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ by telescoping.

6.11. Determine for which real r the geometric series $\sum_{n=0}^{\infty} r^n$ converges.

6.12. Suppose $a_n \geq 0$. Prove $\sum a_n$ converges if and only if its partial sums are bounded above.

6.13. Show that $\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$ diverges. Explain why $a_n \to 0$ is not sufficient for convergence.

6.14. Construct a convergent series with infinitely many negative terms and infinitely many positive terms.

6.15. Let $s_n = 1 - 1/(n+1)$. Find a_n such that $s_n = \sum_{k=1}^{n} a_k$, and evaluate $\sum a_n$.

7 Positive series, comparison and absolute convergence

7.1. Define absolute and conditional convergence. Prove that absolute convergence implies convergence.

7.2. Give an example of a conditionally convergent series and explain both claims.

7.3. State and prove the direct comparison test for nonnegative series.

7.4. Prove the limit comparison test for positive series when the ratio tends to a finite, strictly positive limit.

7.5. Determine convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ by comparison or telescoping.

7.6. For which $p \in \mathbb{R}$ does $\sum_{n=1}^{\infty} n^{-p}$ converge? Give a proof using an appropriate test.

7.7. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges.

7.8. Determine convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

7.9. If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, prove $\sum a_n$ converges. Does the conclusion hold without nonnegativity?

7.10. True or false: if $\sum |a_n|$ diverges, then $\sum a_n$ diverges. Justify.

7.11. If $\sum |a_n|$ converges and (b_n) is bounded, prove $\sum a_n b_n$ converges absolutely.

7.12. Determine convergence of $\sum_{n=1}^{\infty} \frac{n+1}{n^3+1}$.

7.13. Suppose $a_n \geq 0$ and $\sum a_n$ converges. Must $na_n \to 0$? Prove or construct a counterexample.

7.14. Prove that if $\sum a_n$ converges absolutely, then every subseries $\sum_{k=1}^{\infty} a_{n_k}$ converges absolutely.

7.15. If $\sum a_n$ and $\sum b_n$ converge absolutely, prove $\sum(a_n+b_n)$ converges absolutely. Does absolute convergence of $\sum(a_n + b_n)$ imply absolute convergence of both original series?

8 Ratio, root, integral and refined convergence tests

8.1. State and prove the ratio test for a series of nonnegative terms when $\lim a_{n+1}/a_n = L$ exists.

8.2. State the root test using $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$ and prove its convergence assertion.

8.3. Use the ratio test to decide convergence of $\sum_{n=1}^{\infty} n!/n^n$.

8.4. Determine convergence of $\sum_{n=1}^{\infty} n^3/2^n$.

8.5. Determine convergence of $\sum_{n=1}^{\infty}(n/(n+1))^{n^2}$ using the root test.

8.6. Give two series for which the ratio test yields the limiting value 1, one convergent and one divergent.

8.7. Give two series for which the root test yields the limiting value 1, one convergent and one divergent.

8.8. State and prove the integral test for a positive, decreasing function on $[1, \infty)$.

8.9. Determine for which $p, q \in \mathbb{R}$ the series $\sum_{n=2}^{\infty} 1/[n^p(\log n)^q]$ converges.

8.10. Prove Cauchy's condensation test for decreasing positive sequences.

8.11. Apply condensation to $\sum_{n=2}^{\infty} 1/[n(\log n)^p]$.

8.12. Show that $\sum_{n=1}^{\infty}(1 + 1/n)^{-n^2}$ converges.

8.13. Determine convergence of $\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n+n^2}$.

8.14. Explain why $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$ implies absolute convergence, but $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$ gives no general conclusion.

8.15. Find a series with positive terms for which the ratio a_{n+1}/a_n has no limit but the root test proves convergence.

9 Alternating series, Dirichlet and Abel tests

9.1. State and prove the alternating series test, including an estimate for the remainder.

9.2. Prove that $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges but not absolutely.

9.3. Determine convergence type of $\sum_{n=1}^{\infty}(-1)^{n-1}/\sqrt{n}$.

9.4. Does $a_n \to 0$ suffice for convergence of $\sum(-1)^n a_n$ when $a_n \geq 0$ but is not decreasing? Construct a counterexample.

9.5. Prove Dirichlet's test using summation by parts.

9.6. Prove Abel's test for $\sum a_n b_n$ when $\sum a_n$ converges and (b_n) is monotone and bounded.

9.7. Determine convergence of $\sum_{n=1}^{\infty} \sin(n\theta)/n$ for $\theta \notin 2\pi\mathbb{Z}$.

9.8. Prove that if the partial sums of $\sum a_n$ are bounded and $b_n \downarrow 0$, then $\sum a_n b_n$ converges.

9.9. Give an example where Dirichlet's test proves convergence but absolute convergence fails.

9.10. Determine convergence of $\sum_{n=2}^{\infty} (-1)^n / \log n$.

9.11. Find a convergent series $\sum a_n$ and a bounded sequence (b_n) such that $\sum a_n b_n$ diverges. Explain why Abel's monotonicity condition matters.

9.12. Suppose $a_n \geq 0$, $a_n \to 0$, and a_n is eventually decreasing. Prove $\sum(-1)^n a_n$ converges.

9.13. Show that the alternating-series remainder has the sign of the first omitted term under the usual monotonicity assumptions.

9.14. Determine convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ and classify it as absolute or conditional.

9.15. Explain, with an example, why the convergence of $\sum a_n$ does not imply convergence of $\sum |a_n|$.

10 Rearrangements and synthesis

10.1. Define a rearrangement of a series. Prove that rearrangements of absolutely convergent real series have the same sum.

10.2. State the Riemann rearrangement theorem for conditionally convergent real series and outline the role of positive and negative parts.

10.3. For a conditionally convergent series $\sum a_n$, prove that the sums of its positive terms and of the absolute values of its negative terms both diverge to $+\infty$.

10.4. Explain how to rearrange the alternating harmonic series to converge to a prescribed real number T. Give a precise algorithm.

10.5. True or false: every convergent series has the same sum under every rearrangement. Justify.

10.6. If $\sum a_n$ converges absolutely and (b_n) is bounded, prove $\sum a_n b_n$ converges absolutely. Can absolute convergence of $\sum a_n$ be dropped?

10.7. Find convergent series $\sum a_n$ and $\sum b_n$ for which $\sum a_n b_n$ diverges.

10.8. Can $\sum a_n$ converge while $\sum a_n^2$ diverges? Either construct an example or prove impossibility.

10.9. Can $\sum a_n^2$ converge while $\sum a_n$ diverges? Give examples illustrating distinct behaviors.

10.10. Suppose $\sum a_n$ converges and $a_n \geq 0$. Prove $a_n \to 0$ and explain why the converse is false.

10.11. Construct a bounded nonconvergent sequence (x_n) for which $x_{n+1} - x_n \to 0$.

10.12. Prove that if (x_n) is bounded and every convergent subsequence has limit L, then $x_n \to L$.

10.13. Suppose $\sum |a_n| < \infty$ and $x_n \to x$. Prove $\sum a_n x_n$ converges and compare its value with $x \sum a_n$; must they be equal?

10.14. For $a_n = (-1)^{n-1}/n$, investigate the convergence of $\sum a_n$, $\sum |a_n|$, $\sum a_n^2$, and $\sum n a_n$.

10.15. Write a proof or counterexample for each assertion: (a) every Cauchy sequence is bounded; (b) every bounded series of partial sums converges; (c) every bounded sequence has a convergent subsequence; (d) every conditionally convergent series can be rearranged to diverge to $+\infty$.

Selected hints (consult only after attempting the problems)

Section 1. 1.10: Rationalize. 1.12: Solve $2^{-n} < \varepsilon$ for n. 1.13: Define odd terms separately. 1.15: Use the gap between 0 and 1. **Section 2.** 2.6: Rationalize, then divide by n. 2.9: Squaring loses sign information. 2.14: Choose an integer $m > p$ and compare n^p with n^m; use a binomial estimate for c^n. 2.15: Try opposite oscillating sequences. **Section 3.** 3.4: First show invariance of $[1, 2]$. 3.5: The arithmetic–geometric mean inequality helps after the first step. 3.10: Estimate $|x_{n+1} - L|$ or $|x_{n+2} - x_{n+1}|$. 3.11: Factor $x(2 - x) - x = x(1 - x)$. **Section 4.** 4.6: A unique cluster point does not control unbounded terms. 4.7: Use the "peak index" argument. 4.12: Choose indices from increasingly late tails. 4.13: Enumerate rational numbers in $[0, 1]$ in finite stages, or use grids. **Section 5.** 5.5: Compare $H_{2n} - H_n$. 5.6: Sum successive differences. 5.13: Convert the given one-sided inequality into the Cauchy condition. 5.14: Rational truncations converge in $\mathbb{R}$, not in $\mathbb{Q}$. **Section 6.** 6.2: Group terms from $2^k + 1$ through 2^{k+1}. 6.10: Use partial fractions. 6.12: The partial sums are increasing. 6.15: Set $a_1 = s_1$ and $a_n = s_n - s_{n-1}$. **Section 7.** 7.6: Separate $p \leq 0$ and $p > 0$; then use the integral test or condensation. 7.9: The nonnegativity is essential. 7.13: Place rare spikes at powers of 2. 7.15: Consider cancellation between two non-absolutely convergent series. **Section 8.** 8.5: Take the nth root first. 8.9: Compare with the integral of $x^{-p}(\log x)^{-q}$ and separate $p = 1$. 8.10: Compare blocks $[2^k, 2^{k+1})$. 8.15: Try terms with alternating geometric decay rates. **Section 9.** 9.4: Introduce occasional large positive terms at one parity. 9.7: Bound partial sums of $\sin(n\theta)$ using complex exponentials. 9.11: Choose b_n to remove the alternating signs. 9.13: Compare the sum with the next omitted term. **Section 10.** 10.3: Write $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$. 10.7: Try $a_n = b_n = (-1)^n/\sqrt{n}$. 10.8: Try $a_n = (-1)^{n-1}/\sqrt{n}$. 10.12: Argue by contradiction and extract a subsequence staying away from L. 10.13: Separate $x_n = x + (x_n - x)$.