Set Theory Exercise Sheet
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1 Sets

1. Prove the following:
   (a) $(\mathbb{Q} \times \mathbb{Q}^c) \cap (\mathbb{Q}^c \times \mathbb{Q}) = \phi$
   (b) $(\mathbb{Q}^c \times \mathbb{Q}^c) \subset (\mathbb{R} \times \mathbb{Q}^c)$
2. Draw the following sets :
   (a) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\} \cap \{(x, y) \in \mathbb{R} \times \mathbb{R} : (x - 2)^2 + y^2 = 1\}$
   (b) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\} \cap \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y\} \cap \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = 0\} \cap \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 0\}$
3. I'd like to introduce the concept of *path connectedness* to you, not through a dry definition but with a simple story. Many of you may have heard the famous tale of *Nakula* from the *Mahabharatha*. He had the ability to ride his horse through the rain without ever getting wet. In other words, he could find a *path* in the *complement of rainfall*.

   Now, imagine the ground on which *Nakula* rode as a subset of $\mathbb{R} \times \mathbb{R}$, and the falling raindrops as points of $\mathbb{Q} \times \mathbb{Q}$. *Nakula*'s challenge was to find a *path* in the complement of $\mathbb{Q} \times \mathbb{Q}$. Put simply, the task is to show that the complement of $\mathbb{Q} \times \mathbb{Q}$ is *path connected*.

   At our level of mathematics, this means we must be able to draw straight line segments in the complement of $\mathbb{Q} \times \mathbb{Q}$ that connect any two of its points. The question is—can you do it?
4. **Barber's Paradox:** In a certain village, there is a barber who shaves all and only those men in the village who do not shave themselves.

   The question is: **Who shaves the barber?**

   - If the barber shaves himself, then by the rule he should not shave himself.
   - If the barber does not shave himself, then by the rule he must shave himself.

   Either way, we get a contradiction.

   Connection to Russell's Paradox: This story is really just a simple version of *Russell's Paradox* in set theory.

   Instead of barbers and shaving, Russell asked:

   $$R = \{ S : S \notin S \}$$

   the set of all sets that do not contain themselves.

   Now ask: does R contain itself?

   - If $R \in R$, then by definition $R \notin R$.
   - If $R \notin R$, then by definition $R \in R$.

   Again, both answers lead to a contradiction—just like with the barber.

   Why It Is Important in Set Theory: The importance of this paradox is that it shows that **not every condition we can describe actually defines a valid set**.

   - In everyday language, "the barber shaves all those who do not shave themselves" sounds fine—but when analyzed carefully, it breaks down.
   - Similarly, in mathematics, if we allow *any* property to define a set, we run into contradictions like Russell's Paradox.

   This is why modern set theory (such as *Zermelo–Fraenkel set theory*) uses strict axioms to avoid such problems. The paradox teaches us that we must be **careful and precise** when building the foundations of mathematics.

   Conclusion: The Barber's Paradox is more than just a fun story—it is a gateway to understanding why set theory had to be rebuilt on solid axiomatic foundations.

2 Problems on Set Operations

1. Let $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}$.
   (a) Find $A \cup B, A \cap B, A - B, B - A$.
   (b) Verify that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.
2. Let $U = \{1, 2, 3, \ldots, 10\}, A = \{2, 4, 6, 8, 10\}, B = \{1, 2, 3, 4, 5\}$. Find A', B', and $(A \cup B)'$, where X' denotes complement of X in U.
3. If $A = \{x \in \mathbb{Z} : -3 \leq x \leq 3\}$, $B = \{x \in \mathbb{Z} : x^2 \leq 4\}$, find $A \cap B$, $A - B$, $B - A$.
4. Verify the distributive law of sets:
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
   using $A = \{1, 2, 3, 4\}, B = \{2, 4, 6\}, C = \{1, 3, 5\}$.
5. If $U = \{a, b, c, d, e, f\}, A = \{a, b, c\}, B = \{b, c, d, e\}$, find
   (a) $(A \cup B)'$
   (b) $(A' \cap B) \cup (A \cap B')$.
6. Draw a Venn diagram and shade the region representing $(A \cup B) \cap C'$.
7. Let $A = \{x : x \text{ is a prime number less than } 20\}$, $B = \{x : x \text{ is an odd number less than } 20\}$. Find $A \cap B$, $A \cup B$, and $B - A$.
8. For sets A, B, C, show by an example that
$$(A - B) - C \neq A - (B - C).$$
9. Let $A = \{1, 2, 3, 4, 5\}, B = \{3, 4, 5, 6, 7\}$. Find
   (a) $(A \cup B) - (A \cap B)$
   (b) Compare your answer with $(A - B) \cup (B - A)$.
10. Let $U = \{1, 2, \ldots, 12\}, A = \{2, 4, 6, 8, 10, 12\}, B = \{3, 6, 9, 12\}$. Find $(A \cup B)'$ and $(A' \cap B')$. Are they equal?
11. Verify De Morgan's law:
$$(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'$$
   for $U = \{1, 2, 3, 4, 5\}, A = \{1, 2, 3\}, B = \{3, 4\}$.
12. Let $A = \{x \in \mathbb{N} : x < 10, x \text{ is a multiple of } 2\}$, $B = \{x \in \mathbb{N} : x < 10, x \text{ is a multiple of } 3\}$. Find $A \cup B$, $A \cap B$, $A - B$.
13. Draw Venn diagrams to illustrate the following identities:
   (a) $A - (B \cup C) = (A - B) \cap (A - C)$
   (b) $(A \cup B) - C = (A - C) \cup (B - C)$.
14. Let $A = \{1, 2, 3, 4, 5\}, B = \{2, 4, 6, 8\}, C = \{1, 3, 6, 7\}$. Find $(A \cap B) \cup (B \cap C) \cup (C \cap A)$.
15. Suppose A, B are subsets of a universal set U. Prove or disprove:
$$(A - B)' = A' \cup B.$$
   (Hint: Use Venn diagram or algebraic reasoning.)

3 Problems Sheet Cartesian Products and Relations

1. If $A = \{1, 2\}, B = \{a, b\}$, find $A \times B$ and $B \times A$. Are they equal?
2. Let $A = \{x, y\}, B = \{1, 2, 3\}$. Find
   (a) $|A \times B|$
   (b) $|B \times A|$
   (c) Compare both.
3. If $A = \{1, 2, 3\}$, list all ordered pairs in $A \times A$.
4. Let $A = \{a, b\}, B = \{1, 2\}, C = \{x\}$. Verify that
   $$A \times (B \times C) \neq (A \times B) \times C.$$
5. For $A = \{1, 2, 3\}, B = \{2, 4\}$, find
   (a) $A \times B$
   (b) $(A \cup B) \times A$
6. If $A = \{1, 2\}, B = \{3, 4\}$, verify that
   $$(A \cup B) \times (A \cap B) = \emptyset.$$
7. Let $A = \{1, 2, 3, 4\}$. Define a relation R on A by
   $$R = \{(x, y) \in A \times A : x < y\}.$$
   List all ordered pairs in R.
8. For the set $A = \{1, 2, 3\}$, define a relation $R = \{(x, y) : x + y \text{ is even}\}$. Find R explicitly.
9. Let $A = \{a, b, c\}$, and let $R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$. Check if R is:
   (a) reflexive
   (b) symmetric
   (c) transitive
10. On $A = \{1, 2, 3\}$, consider the relation
    $$R = \{(1, 2), (2, 3), (1, 3)\}.$$
    Is R transitive? Reflexive? Symmetric?

11*. **Definition:** A relation R on a set A is called an *equivalence relation* if it is
   (a) reflexive: $(a, a) \in R$ for all $a \in A$,
   (b) symmetric: $(a, b) \in R \implies (b, a) \in R$,
   (c) transitive: $(a, b), (b, c) \in R \implies (a, c) \in R$.

   Define relation R on $\mathbb{Z}$ by $xRy \iff x - y$ is even. Show that R is an equivalence relation. *(This problem is starred as it is not part of the syllabus.)*

12*. Define relation R on $\mathbb{Z}$ by $xRy \iff x \leq y$. Show that R is a partial order but not an equivalence relation.

13*. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define relation $R = \{(x, y) : x \text{ divides } y\}$. List R and check if it is reflexive, antisymmetric, and transitive.

14. Let $A = \{1, 2, 3\}$. How many possible relations can be defined on A? (Hint: Count subsets of $A \times A$.)

15*. Let $A = \{1, 2, 3, 4\}$. Define relation $R = \{(x, y) : |x - y| \leq 1\}$. Write down R and check if it is symmetric and transitive.

4 Problems on Functions

1. Define a function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x + 3$. Find $f(0), f(2), f(-1)$.
2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x^2$. Find the images of $-2, 0, 3$.
3. Find the domain and range of the following functions:
   (a) $f(x) = \sqrt{x}$
   (b) $g(x) = \dfrac{1}{x-2}$
   (c) $h(x) = |x|$
4. Let $f : \{1, 2, 3\} \to \{a, b\}$ be defined by $f(1) = a, f(2) = b, f(3) = a$. Is f one–one? Is it onto?
5. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Is f one–one? Is it onto? Justify.
6. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x + 1$. Show that f is bijective.
7. Let $f : \{1, 2, 3, 4\} \to \{a, b, c, d\}$ be defined by $f(1) = a, f(2) = b, f(3) = c, f(4) = d$. Is f bijective?
8. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = x^2$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.
9. If $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + 1$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = 2x$, compute $(g \circ f)(2)$ and $(f \circ g)(2)$.
10. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x + 5$. Find the inverse function f^{-1}. Verify that $f(f^{-1}(x)) = x$.
11. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Show that f is invertible and find f^{-1}.
12. For $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|$, determine whether f is invertible. If not, restrict the domain suitably to make it invertible, and find the inverse on that domain.
13. If $f : \{1, 2, 3\} \to \{a, b\}$, how many distinct functions are possible? (Hint: Count possible images of each element.)
14. Prove that the composition of two bijections is also a bijection.
15. Define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(x) = x + 1$. Show that f is bijective and find its inverse.

5 Functions and Set Size (Cardinality) $\implies$ Comparing Sizes

0.1 Introduction and Formal Definitions

The properties of **one-to-one** and **onto** functions are the fundamental tools we use to compare the **size** of different sets.

0.1.1 What is Cardinality?

The **cardinality** of a set, denoted as $|A|$, is the number of elements in that set. For the first part of this lecture, we focus only on **finite sets** (sets whose elements can be counted).

- **One-to-One Function (Injective)**: A function $f : A \to B$ is one-to-one if distinct elements in the domain map to distinct elements in the codomain.

$$\text{Formally: If } f(a_1) = f(a_2), \text{ then } a_1 = a_2 \quad \text{for all } a_1, a_2 \in A.$$

- **Onto Function (Surjective)**: A function $f : A \to B$ is onto if every element in the codomain B is the image of at least one element in the domain A.

$$\text{Formally: For every } b \in B, \text{ there exists an } a \in A \text{ such that } f(a) = b.$$

0.2 Cardinality Rules for Finite Sets

Let A and B be two finite sets with a function $f : A \to B$. Let $|A| = n$ and $|B| = m$.

0.2.1 If f is One-to-One (Injective)

Since every element in the domain A must map to a **unique** element in B, B must have at least as many elements as A to avoid sharing outputs.

Result 1: If f is one-to-one, then the codomain is larger (or equal) to the domain.

$$|B| \geq |A| \quad \text{or} \quad m \geq n$$

0.2.2 If f is Onto (Surjective)

Since every element in the codomain B must be hit (used as an output), and we only have n inputs in A, B cannot have more elements than A.

Result 2: If f is onto, then the domain is larger (or equal) to the codomain.

$$|A| \geq |B| \quad \text{or} \quad n \geq m$$

0.3 The Cardinality Equivalence: Bijections

0.3.1 Definition: Bijective Function (One-to-One Correspondence)

A function $f : A \to B$ is a **bijection** if it is **both** one-to-one (injective) and onto (surjective).

0.3.2 The Fundamental Theorem for Finite Sets

For a function $f : A \to B$ to be **both** one-to-one (requiring $|B| \geq |A|$) and onto (requiring $|A| \geq |B|$), the only possibility is that their cardinalities are exactly equal.

Core Principle: Two finite sets A and B have the same cardinality
if and only if there exists a bijection $f : A \to B$.

$$f \text{ is a bijection} \quad \iff \quad |A| = |B|$$

0.3.3 Example: Demonstrating Bijection

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. $|A| = 3$ and $|B| = 3$. Define $f : A \to B$ by $f(x) =$ (the x-th letter in B).

$$f(1) = a, \quad f(2) = b, \quad f(3) = c$$

- **One-to-One?**: Yes. $1 \neq 2$, and $f(1) = a \neq f(2) = b$. Unique inputs $\implies$ unique outputs.
- **Onto?**: Yes. The outputs $\{a, b, c\}$ are exactly equal to the codomain B.

Conclusion: f is a bijection, confirming $|A| = |B| = 3$.

0.4 The Infinite Cardinality Surprise! $\aleph_0$

The idea of using a bijection to define "same size" is so fundamental that we extend it to infinite sets. When dealing with infinite sets, we use the phrase **same cardinality** instead of "same number of elements."

0.4.1 Natural Numbers $\mathbb{N}$ and a Proper Subset

Let $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$. The cardinality of $\mathbb{N}$ is denoted $\aleph_{\mathbf{0}}$ (Aleph-null).

Consider the set $E = \{2, 4, 6, 8, \dots\}$ (the set of even natural numbers). Notice that E is a **proper subset** of $\mathbb{N}$ ($1 \in \mathbb{N}$ but $1 \notin E$). It seems "smaller."

Define the function $f : \mathbb{N} \to E$ by:

$$f(x) = 2x$$

- **One-to-One?**: If $f(a) = f(b)$, then $2a = 2b$, which implies $a = b$. (✓Yes)
- **Onto?**: For any even number $y \in E$, y is of the form $2k$. The input needed is $x = k$. Since $k \in \mathbb{N}$, the function is onto. (✓Yes)

The Revelation: f is a **bijection** between $\mathbb{N}$ and E. Therefore, they must have the same size!

$$|\mathbb{N}| = |E| = \aleph_0$$

Key Property of Infinite Sets: An infinite set can be put into a bijection with a **proper subset** of itself. This is the definition that distinguishes infinite sets from finite ones.

0.4.2 Further Wonders

By establishing a bijection, we can show that the set of natural numbers ($\mathbb{N}$), the set of integers ($\mathbb{Z}$), and the set of rational numbers ($\mathbb{Q}$) all have the **same cardinality**, $\aleph_0$. They are all considered the "smallest" type of infinity, or **countably infinite**.

Future Exploration: When we get to the real numbers ($\mathbb{R}$), we will find that no bijection exists from $\mathbb{N}$ to $\mathbb{R}$. This means $|\mathbb{R}| > |\mathbb{N}|$, introducing the concept of a "bigger" infinity, which we call **uncountable**.