Topics In Analysis
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1 Signed Measure

Definition 1. *Let $(X, \mathcal{M})$ be a measurable space. A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that:*

- *$\nu(\varnothing) = 0$;*
- *ν assumes at most one of the values $\pm\infty$;*
- *if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\cup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$, where the latter sum converges absolutely if $\nu(\cup_1^\infty E_j)$ is finite.*

Remark. *Thus every measure is a signed measure; for emphasis we shall sometimes refer to measures as **positive measures**.*

Example 1. *First, if μ_1, μ_2 are measures on $\mathcal{M}$ and at least one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.*

Example 2. *Second, if μ is a measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite (in which case we shall call f an **extended μ-integrable function**), then the set function ν defined by $\nu(E) = \int_E f d\mu$ is a signed measure.*

Remark. *In fact, we shall see shortly that these are really the only examples: Every signed measure can be represented in either of these two forms.*

Proposition. *Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, then $\nu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$. If $\{E_j\}$ is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then $\nu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.*

Proof. Setting $E_0 = \varnothing$, we have

$$\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \sum_1^n \nu(E_j \setminus E_{j-1}) = \lim_{n\to\infty} \nu(E_n).$$

Let $F_j = E_1 \setminus E_j$; then $F_1 \subset F_2 \subset \cdots$, $\nu(E_1) = \nu(F_j) + \nu(E_j)$, and $\bigcup_1^\infty F_j = E_1 \setminus (\bigcap_1^\infty E_j)$. By the previous result, then,

$$\nu(E_1) = \nu\left(\bigcap_1^\infty E_j\right) + \lim_{j\to\infty} \nu(F_j) = \nu\left(\bigcap_1^\infty E_j\right) + \lim_{j\to\infty} [\nu(E_1) - \nu(E_j)].$$

Since $\nu(E_1)$ is finite, we can subtract it from both sides to obtain the desired result. □

Definition 2. *If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called **positive** (resp. **negative**, **null**) for ν if $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0$, $\nu(F) = 0$) for all $F \in \mathcal{M}$ such that $F \subset E$.*

Remark. *Thus, in the example $\nu(E) = \int_E f d\mu$ described above, E is positive, negative, or null precisely when $f \geq 0$, $f \leq 0$, or $f = 0$ μ-a.e. on E.*

Lemma. *Any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive.*

Proof. The first assertion is obvious from the definition of positivity. If $P_1, P_2, \ldots$ are positive sets, let $Q_n = P_n \setminus \bigcup_1^{n-1} P_j$. Then $Q_n \subset P_n$, so Q_n is positive. Hence if $E \subset \bigcup_1^\infty P_j$, then $\nu(E) = \sum_1^\infty \nu(E \cap Q_j) \geq 0$, as desired. □

Theorem 1 (The Hahn Decomposition Theorem)**.** *If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \varnothing$. If P', N' is another such pair, then $P \Delta P'$ $(= N \Delta N')$ is null for ν.*

Proof. Without loss of generality, we assume that ν does not assume the value ∞. (Otherwise, we could consider $-\nu$ instead).

Step 1: Define the supremum m.
Let m be the supremum of $\nu(E)$ as E ranges over all positive sets in $\mathcal{M}$:

$$m = \sup\{\nu(E) : E \text{ is a positive set for } \nu\}$$

Since the empty set $\varnothing$ is always positive and $\nu(\varnothing) = 0$, we know $m \geq 0$. By the definition of a supremum, there exists a sequence $\{P_j\}$ of positive sets such that $\nu(P_j) \to m$.

Step 2: Construct the set P.
Let $P = \bigcup_1^\infty P_j$. According to the **above Lemma**, the union of a countable family of positive sets is positive; therefore, P is a positive set. Furthermore, by the **above Proposition** (continuity from below), we have:

$$\nu(P) = \lim_{n\to\infty} \nu\left(\bigcup_1^n P_j\right)$$

Since P is positive and ν does not take the value ∞, it follows that $\nu(P) = m$, and in particular, $m < \infty$.

Remark. *If $m = \infty$, then $\nu(P) = \infty$. However, we assumed at the outset that ν does not assume the value ∞. Thus, we must have $m < \infty$.*

Step 3: Define the candidate for the negative set N.
We define $N = X \setminus P$. To complete the Hahn Decomposition, we must show that N is a negative set. We will proceed by assuming that N is not negative and attempting to derive a contradiction.

Step 4: N contains no non-null positive sets.
First, we observe that N cannot contain any non-null positive sets. Suppose there exists a set $E \subset N$ such that E is positive and $\nu(E) > 0$. Since E and P are disjoint, their union $E \cup P$ is also a positive set by the **above Lemma**. The measure of this union would be:

$$\nu(E \cup P) = \nu(E) + \nu(P) = \nu(E) + m > m.$$

This is impossible because m is the supremum of measures of all positive sets. Thus, no such E can exist within N.

Step 5: Finding a subset with larger measure.
If $A \subset N$ and $\nu(A) > 0$, there exists a subset $B \subset A$ such that $\nu(B) > \nu(A)$. This is because A cannot be a positive set (otherwise $P \cup A$ would be a positive set with measure greater than m, contradicting the definition of m). Since A is not positive, there must exist some $C \subset A$ with $\nu(C) < 0$. By defining $B = A \setminus C$, we remove the negative portion, and by additivity:

$$\nu(B) = \nu(A) - \nu(C) > \nu(A).$$

Step 6: Inductive construction of the sequences $\{A_j\}$ and $\{n_j\}$.
If N is not negative, we can construct a sequence of subsets $\{A_j\}$ of N and a sequence of positive integers $\{n_j\}$ to track the "improvement" in measure.

- **Base Case:** Let n_1 be the smallest positive integer such that there exists a set $B \subset N$ with $\nu(B) > n_1^{-1}$. We then pick one such set and call it A_1.
- **Inductive Step:** Once we have A_{j-1}, we look for the smallest positive integer n_j such that there exists a set $B \subset A_{j-1}$ with:
$$\nu(B) > \nu(A_{j-1}) + n_j^{-1}$$
We then pick one such set and call it A_j.

This construction ensures that $A_1 \supset A_2 \supset A_3 \ldots$ and that at each step, we are choosing a subset that increases the measure by at least $1/n_j$, where n_j is as small as possible.

Step 7: The Final Contradiction.
Let $A = \bigcap_1^\infty A_j$. By the continuity of the measure, we have:

$$\infty > \nu(A) = \lim_{j\to\infty} \nu(A_j) > \sum_1^\infty n_j^{-1}$$

For this sum to converge, we must have $n_j \to \infty$ as $j \to \infty$.

However, since N is assumed to be non-negative, the same logic from Step 4 applies to A: there exists a subset $B \subset A$ with $\nu(B) > \nu(A) + n^{-1}$ for some integer n. For a sufficiently large j, we will eventually have $n < n_j$. Since $B \subset A \subset A_{j-1}$, this n would have been a smaller integer than n_j that satisfied the condition for A_j. This contradicts the fact that n_j was chosen as the *smallest* such integer.

Thus, the assumption that N is not negative is untenable. We conclude N is negative, completing the decomposition $X = P \cup N$.

Step 8: Uniqueness of the Decomposition.
Finally, we address the uniqueness of the sets P and N. Suppose P', N' is another pair satisfying the theorem. We consider the difference $P \setminus P'$.

- Since $P \setminus P' \subset P$, it must be a positive set.
- Since $P \setminus P' \subset N'$, it must be a negative set.

A set that is both positive and negative for ν is necessarily a **null** set. By the same logic, $P' \setminus P$ is also null. Therefore, the symmetric difference $P \Delta P' = (P \setminus P') \cup (P' \setminus P)$ is null for ν. Since $N \Delta N'$ is identical to $P \Delta P'$, the decomposition is unique up to null sets. □

Definition 3. *The representation of X as the disjoint union $X = P \cup N$, where P is a positive set and N is a negative set, is called a* ***Hahn decomposition*** *for the signed measure ν.*

Remark. *A Hahn decomposition is usually not unique. For instance, ν-null sets can be transferred from P to N (or vice-versa) without changing the properties of the decomposition. Despite this slight variation in the sets themselves, the decomposition is vital because it leads to a canonical representation of ν as the difference of two positive measures.*

Definition 4. *Two signed measures μ and ν on $(X, \mathcal{M})$ are said to be* ***mutually singular*** *(or ν is* ***singular with respect to*** *μ) if there exist disjoint sets $E, F \in \mathcal{M}$ such that:*

$$E \cup F = X, \quad E \cap F = \varnothing,$$

where E is null for μ and F is null for ν.

Informally, mutual singularity means that μ and ν "live on disjoint sets". We express this relationship symbolically as:

$$\mu \perp \nu.$$

Theorem 2 (The Jordan Decomposition Theorem)**.** *If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. **Step 1: Definition of ν^+ and ν^- and the difference.**
Let $X = P \cup N$ be a Hahn decomposition for ν. We define the positive and negative variations of ν as follows:

$$\nu^+(E) = \nu(E \cap P) \quad \text{and} \quad \nu^-(E) = -\nu(E \cap N)$$

By these definitions, it is clear that $\nu = \nu^+ - \nu^-$. Furthermore, since ν^+ is supported on P and ν^- is supported on N, and $P \cap N = \varnothing$, we have $\nu^+ \perp \nu^-$.

Step 2: Uniqueness and the alternative split.
Suppose there is another pair of positive measures μ^+ and μ^- such that $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. Because they are mutually singular, there exist disjoint sets E and F such that $E \cup F = X$, with $\mu^+(F) = 0$ and $\mu^-(E) = 0$. This implies that $X = E \cup F$ is another valid Hahn decomposition for ν.

Step 3: Proving $\mu^+(A) = \nu^+(A)$ and Conclusion.
Since $E \cup F$ and $P \cup N$ are both Hahn decompositions, the symmetric difference $P \Delta E$ is ν-null. Therefore, for any measurable set A:

$$\mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$$

By the same logic, $\mu^- = \nu^-$. This confirms that the decomposition $\nu = \nu^+ - \nu^-$ is unique. □

Definition 5. *The positive measures ν^+ and ν^- are called the **positive and negative variations** of ν, respectively. The unique representation $\nu = \nu^+ - \nu^-$ is called the **Jordan decomposition** of ν.*

Definition 6. *We define the **total variation** of the signed measure ν to be the positive measure $|\nu|$ given by:*

$$|\nu| = \nu^+ + \nu^-$$

Remark: This is analogous to how a function of bounded variation on $\mathbb{R}$ can be written as the difference between two increasing functions.

Exercise 1. *Let ν and μ be signed measures on a measurable space $(X, \mathcal{M})$, and let $|\nu| = \nu^+ + \nu^-$ be the total variation of ν. Verify the following properties:*

1. *$E \in \mathcal{M}$ is ν-null iff $|\nu|(E) = 0$.*
2. *$\nu \perp \mu$ iff $|\nu| \perp \mu$.*
3. *$\nu \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Solution. **1. Null Sets:**
A set E is ν-null if every measurable subset of E has a measure of zero.

- Assume $|\nu|(E) = 0$. Since $|\nu| = \nu^+ + \nu^-$ and both ν^+, ν^- are positive measures, it must be that $\nu^+(E) = 0$ and $\nu^-(E) = 0$. For any $A \subset E$, $|\nu(A)| \leq \nu^+(A) + \nu^-(A) \leq 0$, so $\nu(A) = 0$. Thus, E is ν-null.
- Conversely, if E is ν-null, then by definition $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = -\nu(E \cap N) = 0$. Therefore, $|\nu|(E) = 0 + 0 = 0$.

2. Singularity of Total Variation:
If $\nu \perp \mu$, there exist disjoint sets A, B such that $A \cup B = X$, where A is ν-null and B is μ-null. From Part 1, we know A is ν-null if and only if $|\nu|(A) = 0$. Since A is $|\nu|$-null and B is μ-null, the condition for $|\nu| \perp \mu$ is satisfied.

3. Individual Components:

- **Forward Direction:** If $|\nu| \perp \mu$, there is a set A where $|\nu|(A) = 0$ and $\mu(A^c) = 0$. Since $0 \leq \nu^+ \leq |\nu|$ and $0 \leq \nu^- \leq |\nu|$, it follows that $\nu^+(A) = 0$ and $\nu^-(A) = 0$. Thus, $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.
- **Backward Direction:** Suppose $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. There exist sets A, B such that $\nu^+(A) = 0, \mu(A^c) = 0$ and $\nu^-(B) = 0, \mu(B^c) = 0$. Let $C = A \cap B$. Then $|\nu|(C) = \nu^+(C) + \nu^-(C) \leq \nu^+(A) + \nu^-(B) = 0$. Meanwhile, $\mu(C^c) = \mu(A^c \cup B^c) \leq \mu(A^c) + \mu(B^c) = 0$. This proves $|\nu| \perp \mu$.

□

Definition 7. *The representation $X = P \cup N$ of the space as the disjoint union of a positive set P and a negative set N is called a **Hahn decomposition** for the signed measure ν. It is unique up to ν-null sets.*

Definition 8. *Two signed measures μ and ν on $(X, \mathcal{M})$ are **mutually singular** (written $\mu \perp \nu$) if there exist disjoint sets $E, F \in \mathcal{M}$ such that $E \cup F = X$, where E is null for μ and F is null for ν.*

Theorem 3 (3.4 The Jordan Decomposition Theorem)**.** *If ν is a signed measure, there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Remark. *The measures ν^+ and ν^- are called the **positive and negative variations** of ν. We define the **total variation** of ν as the measure $|\nu| = \nu^+ + \nu^-$.*

1.1 Integration and Finiteness

Integration with respect to a signed measure ν is defined by:

$$\int f\,d\nu = \int f\,d\nu^+ - \int f\,d\nu^- \quad \text{for } f \in L^1(\nu)$$

where $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$.

- A signed measure ν is **finite** (or σ-finite) if $|\nu|$ is finite (or σ-finite).
- If ν omits the value ∞, then ν^+ is a finite measure and ν is bounded above.
- If the range of ν is contained in $\mathbb{R}$, then ν is bounded.

Exercise 2. *Verify the following properties for a signed measure ν and its total variation $|\nu|$:*

1. *$E \in \mathcal{M}$ is ν-null iff $|\nu|(E) = 0$.*
2. *$\nu \perp \mu$ iff $|\nu| \perp \mu$.*
3. *$\nu \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Solution. **1. Null Sets:** If $|\nu|(E) = 0$, then $\nu^+(E) = \nu^-(E) = 0$. For any $A \subset E$, $|\nu(A)| \leq \nu^+(A) + \nu^-(A) = 0$, so $\nu(A) = 0$ and E is ν-null. Conversely, if E is ν-null, then $\nu^+(E) = \nu(E \cap P) = 0$ and $\nu^-(E) = -\nu(E \cap N) = 0$, so $|\nu|(E) = 0$.

2 & 3. Singularity: Since ν and $|\nu|$ share the same null sets, $\nu \perp \mu$ implies there is a ν-null set A (which is also $|\nu|$-null) whose complement is μ-null, thus $|\nu| \perp \mu$. Because $0 \leq \nu^+, \nu^- \leq |\nu|$, $|\nu| \perp \mu$ implies its components ν^+ and ν^- must also be singular to μ. □

2 The Lebesgue-Radon-Nikodym Theorem

Definition 9. *Suppose ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is **absolutely continuous** with respect to μ, and write $\nu \ll \mu$, if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$.*

Exercise 3. *Show that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.*

Solution. The result follows by utilizing the definitions of the Jordan decomposition $\nu = \nu^+ - \nu^-$ and the total variation $|\nu| = \nu^+ + \nu^-$.

- If $\mu(E) = 0$, then $\nu \ll \mu \implies \nu(A) = 0$ for all $A \subset E$. Applying this to $A = E \cap P$ and $A = E \cap N$ shows $\nu^+(E) = 0$ and $\nu^-(E) = 0$.
- Conversely, if $\nu^+(E) = 0$ and $\nu^-(E) = 0$, then their sum $|\nu|(E)$ and difference $\nu(E)$ must also be zero.

□

Remark. *Absolute continuity is the antithesis of mutual singularity. Specifically, if $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$. This is verified by noting that if $X = E \cup F$ with $\mu(E) = 0$ and $|\nu|(F) = 0$, then $\nu \ll \mu$ forces $|\nu|(E) = 0$, whence $|\nu| = 0$.*

Theorem 4. *Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. **Step 1: Reduction to the positive case.**
Since $\nu \ll \mu$ iff $|\nu| \ll \mu$ and $|\nu(E)| \leq |\nu|(E)$, it suffices to assume for the remainder of the proof that $\nu = |\nu|$ is a positive measure.

Step 2: The ϵ-δ condition implies absolute continuity.
Clearly, if the ϵ-δ condition is satisfied, then $\mu(E) = 0$ implies $\mu(E) < \delta$ for all $\delta > 0$, which forces $\nu(E) < \epsilon$ for all $\epsilon > 0$. Thus, $\nu(E) = 0$, meaning $\nu \ll \mu$.

Step 3: Setting up the converse by contradiction.
If the ϵ-δ condition is not satisfied, there exists some $\epsilon > 0$ such that for all $n \in \mathbb{N}$, we can find $E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \epsilon$. We define the decreasing sequence of sets:

$$F_k = \bigcup_{n=k}^{\infty} E_n \quad \text{and its intersection} \quad F = \bigcap_{k=1}^{\infty} F_k$$

Step 4: Concluding $\mu(F) = 0$.
By the subadditivity of μ, we have $\mu(F_k) < \sum_{n=k}^{\infty} 2^{-n} = 2^{1-k}$. As $k \to \infty$, it follows that $\mu(F) = 0$.

Step 5: Proving $\nu(F) \geq \epsilon$ and getting the contradiction.
Since $E_k \subset F_k$, we have $\nu(F_k) \geq \epsilon$ for all k. Because ν is a finite measure, we can use continuity from above:

$$\nu(F) = \lim_{k\to\infty} \nu(F_k) \geq \epsilon$$

Thus, we have a set F where $\mu(F) = 0$ but $\nu(F) > 0$, which contradicts the assumption that $\nu \ll \mu$. □

Remark. *The requirement that ν is finite is essential for the "only if" direction. If ν were allowed to be infinite, the limit step in the proof would not necessarily lead to a contradiction in the same manner.*

Exercise 4. *Suppose μ is a measure and f is an extended μ-integrable function. Let the signed measure ν be defined by $\nu(E) = \int_E f\,d\mu$. Show that $\nu \ll \mu$ and that ν is finite iff $f \in L^1(\mu)$.*

Solution. First, we verify absolute continuity. If $E \in \mathcal{M}$ such that $\mu(E) = 0$, then by the properties of integration:

$$\nu(E) = \int_E f\,d\mu = 0$$

which satisfies the definition of $\nu \ll \mu$.

Next, we consider finiteness. Recall that a signed measure ν is finite iff its total variation $|\nu|$ is finite. For the measure defined above, $|\nu|(E) = \int_E |f|\,d\mu$. Thus:

$$|\nu|(X) = \int_X |f|\,d\mu$$

The right-hand side is finite if and only if $f \in L^1(\mu)$. Therefore, ν is a finite signed measure iff $f \in L^1(\mu)$. □

Corollary 5 (3.6)**.** *If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f\,d\mu| < \epsilon$ whenever $\mu(E) < \delta$.*

Proof. Let $\nu(E) = \int_E f\,d\mu$. As shown previously, ν is a finite signed measure because $f \in L^1(\mu)$, and it is absolutely continuous with respect to μ ($\nu \ll \mu$). By Theorem 4, since ν is a finite signed measure, the absolute continuity $\nu \ll \mu$ is equivalent to the stated ϵ-δ condition:

$$\left|\int_E f\,d\mu\right| = |\nu(E)| < \epsilon \quad \text{whenever} \quad \mu(E) < \delta$$

This completes the proof. □

2.1 Notation for Measures as Integrals

When a signed measure ν is related to a positive measure μ via an integrable function f such that $\nu(E) = \int_E f\,d\mu$, we use the differential notation:

$$d\nu = f\,d\mu$$

By a slight abuse of language, we may refer to this relationship simply as "the signed measure $f\,d\mu$."

Lemma. *Suppose that ν and μ are finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$, or there exist $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E (that is, E is a positive set for $\nu - \epsilon\mu$).*

Proof. **Step 1: Defining the sequence of measures σ_n.**
For each $n \in \mathbb{N}$, let σ_n be the signed measure defined by $\sigma_n = \nu - n^{-1}\mu$. Let $X = P_n \cup N_n$ be a Hahn decomposition for σ_n.

Step 2: Defining the sets P and N.
Let $P = \bigcup_1^\infty P_n$ and $N = \bigcap_1^\infty N_n = P^c$.

Step 3: Calculating $\nu(N) = 0$.
By construction, N is a subset of N_n for every n, so N is a negative set for $\nu - n^{-1}\mu$ for all n. This implies $0 \leq \nu(N) \leq n^{-1}\mu(N)$ for all n. Since μ is a finite measure, as $n \to \infty$, we must have $\nu(N) = 0$.

Step 4: The case where $\mu(P) = 0$.
If $\mu(P) = 0$, then since $\nu(N) = 0$ and $P \cup N = X$ with $P \cap N = \emptyset$, we satisfy the definition of mutual singularity: $\nu \perp \mu$.

Step 5: The case where $\mu(P) > 0$.
If $\mu(P) > 0$, then by the properties of measures, $\mu(P_n) > 0$ for some n. For this n, let $E = P_n$ and $\epsilon = n^{-1}$. Then $\mu(E) > 0$ and E is a positive set for $\nu - \epsilon\mu$, which completes the proof. ☐